The ∇ Operator

What is a Vector?

If S and S' are two sets of Cartesian coordinates sharing a common origin, then vector **A**, which has components A_x, A_y, A_z in the S frame has components in the S' frame given by

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

Where the coefficients $\{a_{ij}\}$ depend only on the orientation of S' with respect to S. If a quantity transforms in this way then it is a vector, otherwise it is not.

The ∇ Operator

If we let $\mathbf{A} = \nabla f$ then provided f is differentiable

$$A_x = \frac{\partial f}{\partial x} \quad \text{and} \quad A_{x'} = \frac{\partial f}{\partial x'} = a_{xx}\frac{\partial f}{\partial x} + a_{xy}\frac{\partial f}{\partial y} + a_{xz}\frac{\partial f}{\partial z}$$

with similar result for the other two components so ∇f is certainly a vector. We can also see from this that the components of ∇ which were defined in the S frame are given in the S' frame by

$$\nabla_{x'} = \frac{\partial}{\partial x'} = a_{xx}\frac{\partial}{\partial x} + a_{xy}\frac{\partial}{\partial y} + a_{xz}\frac{\partial}{\partial z} \quad etc.$$

so the operator ∇ transforms in the required way and is, therefore a vector in its own right. However it is also a differential operator and so ∇ must only be used in ways that satisfy simultaneously the rules for manipulating vectors, and of partial differentiation.