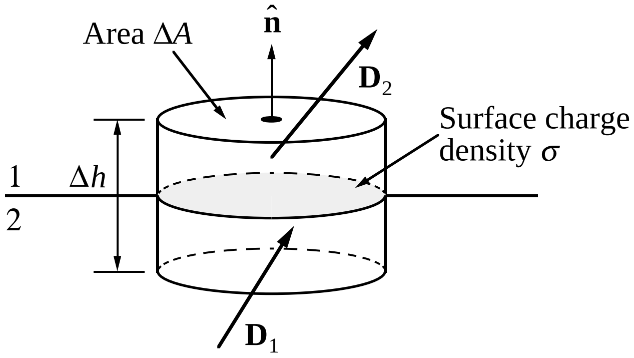
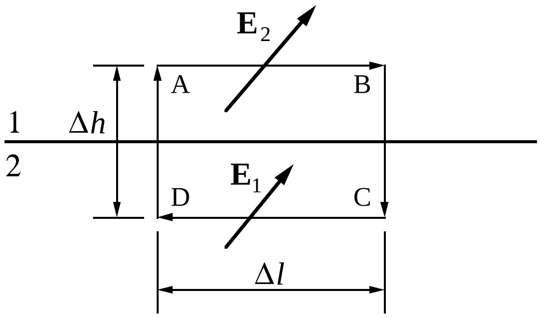
Electrostatic Boundary Conditions





Consider a Gaussian pill-box at the interface between two different media, arranged as in the figure above. The net enclosed (free) charge Q_f is

$$Q_\mathrm{f} = \sigma \Delta A + \tfrac{1}{2}\left(\rho_1 + \rho_2\right)\Delta A \Delta h$$

so as the height of the pill-box Δh tends to zero the term arising from the bulk charge densities ρ_1, ρ_2 becomes negligible. The integral form of Gauss's law then tells us that

$$\left(\mathbf{D}_2 \cdot \hat{\mathbf{n}}\right)\Delta A - \left(\mathbf{D}_1 \cdot \hat{\mathbf{n}}\right)\Delta A \approx \sigma \Delta A$$

which becomes exact in the limit $\Delta A \to 0$ when

$$\left(\mathbf{D}_2 - \mathbf{D}_1\right) \cdot \hat{\mathbf{n}} = \sigma$$

therefore if there is no free surface charge the component of **D** normal to the interface is continuous.

Since **E** is a conservative field and $\oint_{\mathrm{ABCD}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$ so

$$\text{as } \Delta h \to 0 \quad \text{so} \quad \left(\mathbf{E}_{2\parallel}\Delta l - \mathbf{E}_{1\parallel}\Delta l\right) \to 0$$

and therefore the component of **E** tangential to the interface is continuous across the interface.

There is no reason to suspect that **E** becomes infinite at the boundary and so the potential is continuous across the interface as a consequence of its definition.

Example: A parallel plate capacitor has plates of area *A*, and has the space between them entirely filled by two slabs, also of area *A*, of different dielectric material of thickness *a* and *b* respectively. Ignore edge-effects and find the capacitance of this structure.

Solution: Given the symmetry of the system, and the instruction to ignore edge-effects, the fields have components only normal to the interfaces which are assumed to be horizontal. When there is a potential difference *V* between the top and bottom plate, the work needed per unit charge to move a small test charge between them is

$$aE_{\mathrm{A}} + bE_{\mathrm{B}} = V\,.$$

The component of **D** normal to an interface is continous when there is no *free* charge present, as is the case at the interface between the two types of dielectric, so $D_{\mathrm{A}} = D_{\mathrm{B}}$ and from the definition of relative permittivity

$$D_{\mathrm{A}} = \varepsilon_0 \varepsilon_{\mathrm{rA}} E_{\mathrm{A}} \quad \text{and} \quad D_{\mathrm{B}} = \varepsilon_0 \varepsilon_{\mathrm{rB}} E_{\mathrm{B}}.$$

These equations are sufficient to determine the fields, for example

$$E_A = \frac{\varepsilon_{\mathrm{rB}}}{\varepsilon_{\mathrm{rA}}} E_{\mathrm{B}} \quad \text{so} \quad a\frac{\varepsilon_{\mathrm{rB}}}{\varepsilon_{\mathrm{rA}}} E_{\mathrm{B}} + bE_B = V \quad \Rightarrow \quad E_B = \frac{V\varepsilon_{\mathrm{rA}}}{a\varepsilon_{\mathrm{rB}} + b\varepsilon_{\mathrm{rA}}}.$$

At the interface between the lower metal plate and the dielectric B there is a surface free-charge density σ present causing a discontinuity in **D** given by

$$\left(D_{\mathrm{B}} - D_{\mathrm{Metal}}\right) = \sigma$$

Since when the plates have a vacuum between them $D_{\mathrm{Metal}}=0$ is required give the correct value for the capacitance and we assume that this is generally the case, hence the net charge on the lower plate *Q* is

$$Q = A\left(\varepsilon_0 \varepsilon_{\mathrm{rB}} E_B - 0\right) = \frac{AV\varepsilon_{\mathrm{rA}}\varepsilon_0\varepsilon_{\mathrm{rB}}}{a\varepsilon_{\mathrm{rB}} + b\varepsilon_{\mathrm{rA}}}$$

and the capacitance *C* is therefore

$$C = \frac{Q}{V} = \frac{A\varepsilon_0\varepsilon_{\mathrm{rA}}\varepsilon_{\mathrm{rB}}}{a\varepsilon_{\mathrm{rB}} + b\varepsilon_{\mathrm{rA}}}.$$