Example 2c

Problem

A spherically symmetric charge density $\rho(\mathbf{r}) = \rho_0 \exp(-r/r_0)$ causes an electric field **E**. Find **E** using three different methods:

Direct Integration of Coulomb's Law

$$\mathbf{E}(\mathbf{r}) = \int \frac{\rho(\mathbf{r}').(\mathbf{r}-\mathbf{r}')\,\mathrm{d}^3 r'}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|^3} = \frac{\rho_0}{4\pi\varepsilon_0}\int \frac{\exp(-r'/r_0).(\mathbf{r}-\mathbf{r}')\,\mathrm{d}^3 r'}{|\mathbf{r}-\mathbf{r}'|^3}$$

This can be evaluated in spherical polar coordinates as follows:

$$\mathbf{E}(\mathbf{r}) = \frac{\rho_0}{4\pi\varepsilon_0}\int \frac{\exp(-r'/r_0).\left(\mathbf{r} - r'\left(\hat{\boldsymbol{\theta}}\sin\theta'\cos\varphi' + \hat{\boldsymbol{\varphi}}\sin\theta'\sin\varphi' + \hat{\mathbf{r}}\cos\theta'\right)\right)\sin\theta'\mathrm{d}\theta'\,\mathrm{d}\varphi'\,r'^2\mathrm{d}r'}{\left|\mathbf{r} - r'\left(\hat{\boldsymbol{\theta}}\sin\theta'\cos\varphi' + \hat{\boldsymbol{\varphi}}\sin\theta'\sin\varphi' + \hat{\mathbf{r}}\cos\theta'\right)\right|^3}$$

the components of **E** in the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ directions vanish because of symmetry leaving

$$\mathbf{E}(\mathbf{r}) = \frac{\hat{\mathbf{r}}\rho_0}{4\pi\varepsilon_0}\int \frac{\exp(-r'/r_0).(r - r'\cos\theta')\sin\theta'\mathrm{d}\theta'\,\mathrm{d}\varphi'\,r'^2\mathrm{d}r'}{\left|r^2 + r'^2 - 2rr'\cos\theta'\right|^{3/2}}$$

which is obviously going to be possible, but rather tedious, to evaluate fully.

Integral Form of Gauss's Law

Since the charge distribution is spherically symmetric we can use Gauss's Law to find the field.

$$\mathbf{E}(\mathbf{r}) = \frac{\hat{\mathbf{r}}Q(r)}{4\pi\varepsilon_0 r^2}$$

where $Q(r)$ is the charge enclosed by a sphere of radius r concentric with the charge distribution. So

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{\hat{\mathbf{r}}\rho_0}{4\pi\varepsilon_0 r^2}\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{\pi}\sin\theta'\mathrm{d}\theta'\int_0^{r}\exp(-r'/r_0)\,r'^2\mathrm{d}r' \\ &= \frac{\hat{\mathbf{r}}\rho_0}{\varepsilon_0 r^2}\int_0^{r}\exp(-r'/r_0)\,r'^2\mathrm{d}r'\end{aligned}$$

this form of integral can be evaluated by integrating by parts twice with the result that

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{\hat{\mathbf{r}}\rho_0}{\varepsilon_0 r^2}\int_0^r \exp\left(-r'/r_0\right) r'^2 \mathrm{d}r' = \frac{\hat{\mathbf{r}}\rho_0 r_0^3}{\varepsilon_0 r^2}\int_0^{r/r_0} \exp(-\lambda)\,\lambda^2 \mathrm{d}\lambda \\
&= \frac{\hat{\mathbf{r}}\rho_0 r_0^3}{\varepsilon_0 r^2}\left[-\left(\lambda^2 + 2\lambda + 2\right)\exp(-\lambda)\right]_0^{r/r_0} \\
&= \frac{\hat{\mathbf{r}}2\rho_0 r_0}{\varepsilon_0}\left[\frac{r_0^2}{r^2} - \left(\frac{1}{2} + \frac{r_0}{r} + \frac{r_0^2}{r^2}\right)\exp\left(-r/r_0\right)\right]
\end{aligned}
$$

Differential Form of Gauss's Law

There is no $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\varphi}}$ dependence so $\left(\nabla \cdot \mathbf{E}\right) = \dfrac{\rho}{\varepsilon_0}$ can be written

$$
\frac{1}{r^2}\frac{d}{dr}\left(r^2 E_\mathrm{r}\right) = \frac{\rho}{\varepsilon_0}
$$

which is solved by rearranging and integrating

$$
E_\mathrm{r}(r) - E_\mathrm{r}(0) = \frac{\rho_0}{r^2\varepsilon_0}\int_0^r r'^2 \exp\left(-r'/r_0\right)\mathrm{d}r'.
$$

Since $\mathbf{E}(\mathbf{r}) = \hat{\mathbf{r}}E_\mathrm{r}(r)$ and $E_\mathrm{r}(0) = 0$ this is the same result as was found in the previous section.