Multipole Expansions

The potential $\phi(\mathbf{r})$ due to a distribution of charge density $\rho(\mathbf{r}')$ within a region V is

$$\phi(\mathbf{r}) = \int_{\mathrm{V}} \frac{\rho(\mathbf{r}')\,\mathrm{d}^3 r'}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}$$

which we wish to approximate by a series. There are several possible useful expansions but since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left(1+\left(\frac{r'}{r}\right)^2 - \frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2}\right)^{-1/2}$$

when $r > r'$ a simple binomial series will converge:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'^2}{r^2}-\frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2}\right)+\frac{3}{2}\frac{1}{2}\frac{1}{2!}\left(\frac{r'^2}{r^2}-\frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2}\right)^2+\cdots\right]$$

$$= \frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}-\frac{1}{2}\frac{r'^2}{r^3}+\frac{3}{8r}\cdot\frac{4(\mathbf{r}\cdot\mathbf{r}')^2}{r^4}+\cdots$$

$$= \frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}+\frac{1}{2r^5}\left[3(\mathbf{r}\cdot\mathbf{r}')^2-(\mathbf{r}\cdot\mathbf{r})(\mathbf{r}'\cdot\mathbf{r}')\right]+\cdots$$

$$= \frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}+\frac{1}{2r^5}\sum_{i,j=1}^{3} x_i x_j\left(3x_i' x_j' - \delta_{ij} r'^2\right)+\cdots$$

where x_i (i=1,2,3) are the Cartesian components of $\mathbf{r}$ and δ_{ij} is the Dirac delta function. Therefore

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{\mathbf{r}\cdot\mathbf{p}}{r^3}+\frac{\frac{1}{2}\sum_{ij} x_i x_j Q_{ij}}{r^5}+\cdots\right)$$

where q is the total charge of the source, $\mathbf{p}$ is the dipole moment of the distribution

$$q = \int_{\mathrm{V}} \rho(\mathbf{r}')\,\mathrm{d}^3 r' \qquad \text{and} \qquad \mathbf{p} = \int_{\mathrm{V}} \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}^3 r'$$

and the nine elements of **Q**, the Cartesian quadrupole moment tensor, are defined by

$$Q_{ij} = \int_{\mathrm{V}} \rho(\mathbf{r}')\left(3x_i' x_j' - \delta_{ij} r'^2\right)\mathrm{d}^3 r'.$$

It is easy to show that

$$\mathrm{Tr}(\mathbf{Q}) = Q_{11} + Q_{22} + Q_{33} = 0.$$

Some authors, when the distribution has azimuthal symmetry about the *z*-axis and therefore

$$Q_{33} = -2Q_{11} = -2Q_{22},$$

refer to Q_{33} as "*the* quadrupole moment" which can be a little confusing.

Comments

(i) Note that the expansion is valid only when **r** is outside V.

(ii) For large r usually only the first non-zero term is significant.

(iii) The expansion can equally well be made in terms of spherical harmonics.

(iv) The field of a distribution can be reproduced by placing suitably sized multipoles at the origin.

(v) In general the size of the dipole and higher-order terms depend on the position of the origin.