Analytic Solutions to Laplace's Equation in 2-D

Cartesian Coordinates

When it works, the easiest way to reduce a partial differential equation to a set of ordinary ones is by separating the variables

$$\phi(x,y) = X(x)Y(y) \quad \text{so} \quad \frac{\partial^2\phi}{\partial x^2} = Y(y)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} \quad \text{and} \quad \frac{\partial^2\phi}{\partial y^2} = X(x)\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}$$

$$\Rightarrow \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = 0$$

and since x and y are independent, if this is to be true each term must be a constant and therefore

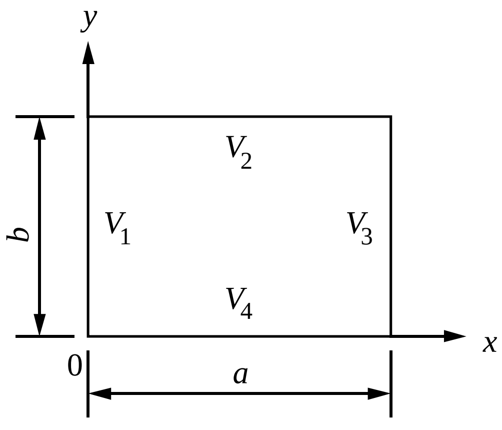
$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = +\lambda^2 X \quad \text{and} \quad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -\lambda^2 Y.$$

So the general solutions have the form

$$Y = \alpha\cos\lambda y + \beta\sin\lambda y \quad \text{and} \quad X = \gamma\cosh\lambda x + \delta\sinh\lambda x$$

with only four of the five constants independent.

Example: Find the potential at an arbitrary point inside a rectangular box of infinite extent in the z-direction with conducting walls at potentials $V_1..V_4$.



Solution: The problem is to choose the value of the constants in the general solution above such that the specified boundary conditions are met. Since the principle of superposition applies to solutions of Laplace's equation let ϕ_1 be the solution when $V_2 = V_3 = V_4 = 0$ so

$$\phi_1(0,y) = V_1 = \gamma(\alpha\cos\lambda y + \beta\sin\lambda y) \Rightarrow \gamma \neq 0$$

$$\phi_1(x,0) = 0 = \alpha(\gamma\cosh\lambda x + \delta\sinh\lambda x) \Rightarrow \text{either } (\gamma = \delta = 0) \text{ or } (\alpha = 0)$$

Since $\gamma = \beta = 0$ is a trivial solution set $\alpha = 0$ and then

$$\phi_1(x,b) = 0 = \beta\sin(\lambda b)(\gamma\cosh\lambda x + \delta\sinh\lambda x) \Rightarrow \lambda = n\pi/b \text{ for integer } n$$

$$\phi_1(a,y) = 0 = \beta \sin\left(\frac{n\pi y}{b}\right)\left[\gamma \cosh\left(\frac{n\pi a}{b}\right) + \delta \sinh\left(\frac{n\pi a}{b}\right)\right] \quad \Rightarrow \quad \delta = -\gamma \coth\left(\frac{n\pi a}{b}\right)$$

$$\phi_1(0,y) = V_1 = \gamma\beta \sin\left(\frac{n\pi y}{b}\right)$$

which is not satisified by any single value of n, so use a Fourier series expansion

$$\phi_1(0,y) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi y}{b}\right) \text{ where } \gamma_n = \frac{2}{a}\int_0^a V_1(y) \sin\left(\frac{n\pi y}{b}\right) \mathrm{d}y$$

and put it all together so that

$$\phi_1(x,y) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi y}{b}\right)\left(\cosh\left(\frac{n\pi x}{b}\right) - \coth\left(\frac{n\pi a}{b}\right)\sinh\left(\frac{n\pi y}{b}\right)\right).$$

The procedure is repeated to find $\phi_2..\phi_4$ and construct the final answer $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. Although the general solution is simple in Cartesian coordinates, getting it to satisfy the boundary conditions can be rather tedious.

Cylindrical Polar Coordinates

In cylindrical polar coordinates when there is no z-dependence $\nabla^2\phi$ has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial r^2}\right) = 0.$$

Separating variables $\phi = R(r)\Theta(\theta)$ so

$$\frac{1}{R}\left(r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\right) = \frac{-1}{\Theta}\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right) = m^2$$

where the positive sign of the constant m^2 anticipates the result

$$\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right) = -m^2\Theta \quad \Rightarrow \quad \Theta = \alpha \cos(m\theta) + \beta \sin(m\theta)$$

as Θ must be periodic so m must be an integer.

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = m^2 R \quad \text{so try} \quad R = Cr^l$$

and $$l^2 Cr^l = l^2 R = m^2 R \quad \Rightarrow \quad l = \pm m.$$

Hence $R=\gamma r^m+\delta r^{-m}$ is the general form for $m\neq 0$ and $R=\alpha_0\ln r+\beta_0$ when $m=0$ and the most general form of the solution is

$$\phi(r,\theta)=\alpha_0\ln r+\beta_0+\sum_{m=1}^{\infty}\left[\left(\gamma_m r^m+\delta_m r^{-m}\right)\left(\alpha_m\cos(m\theta)+\beta_m\sin(m\theta)\right)\right]$$

including a redundant constant.

Example: A long conducting cylinder with its axis along the z-direction is placed into a field $\mathbf{E}=E_0\hat{\mathbf{x}}$. Find the resultant potential if the cylinder radius is a.

Solution: Far enough away from the cylinder the field will be unaffected so

$$\text{as} \quad r\to\infty \quad \mathbf{E}\to E_0\hat{\mathbf{x}} \quad \text{so} \quad \phi\to -E_0x=-E_0\cos(\theta) \quad \Rightarrow \quad \beta_0=0.$$

and symmetry about the x-axis implies that $\phi(\theta)=-\phi(\theta)$ therefore $\beta_m=0$ (for $m>0$). The increasing powers of r would affect the potential at very large distances unless $\gamma_m=0$ (for $m>0$). The required form of the potential is therefore

$$\phi=\alpha_0-E_0r\cos\theta+\sum_{m=1}^{\infty}\frac{\alpha_m}{r^m}\cos(m\theta)$$

where the δ_m have been absorbed into the α_m. On the surface of the conducting cylinder the potential must be constant, say V_0, so

$$\phi(r=a)=V_0 \quad \Rightarrow \quad 0=\left(\alpha_0-V_0\right)+\left(\frac{\alpha_1}{a}-E_0a\right)\cos(\theta)+\sum_{m=2}^{\infty}\left(\frac{\alpha_m}{a^m}\right)\cos(m\theta).$$

Since $\{\cos(m\theta)\}$ are linearly independent functions

$$\alpha_0=V_0, \qquad \alpha_1=E_0a^2, \qquad \alpha_{m\geq 2}=0$$

$$\begin{aligned}\phi(r,\theta)&=V_0-E_0r\cos(\theta)+\frac{E_0a^2}{r}\cos(\theta)\\&=V_0+E_0\left(\frac{a^2}{r}-r\right)\cos(\theta).\end{aligned}$$

Complex Variables

A function $f(z)$ is *analytic* (also known as *regular* or *holomorphic*) if, at a point z_0,

$$\left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z_0}=\lim_{\Delta z\to 0}\frac{f\left(z_0+\Delta z\right)-f\left(z_0\right)}{\Delta z}$$

exists and has a single value. The quantity Δz can be in any direction in the complex plane. If

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

then for Δz along the real axis $\Delta z = \Delta x$ and

$$\frac{\mathrm{d}f}{\mathrm{dz}} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x,y) - v(x,y)]}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

but when Δz is along the imaginary axis $\Delta z = i\Delta y$ and

$$\frac{\mathrm{d}f}{\mathrm{dz}} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the real and imaginary parts of these results yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are both necessary and sufficient conditions for the function f to be analytic.

The Cauchy-Riemann equations can be differentiated

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

and adding the results together gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

because $\dfrac{\partial^2 g}{\partial x \partial y} = \dfrac{\partial^2 g}{\partial y \partial x}$ for a continuous function $g(x,y)$.

Therefore *every* analytic function provides two solutions to Laplace's equation in 2-dimensions, and pairs of such solutions are known as *conjugate harmonic* functions. As the curves u=constant and v=constant are perpendicular to each other, if one represents a contour of constant potential, then the other is a flux line of the corresponding electric field. These interesting properties are the basis of *conformal mapping* methods for solving 2-dimensional electrostatic problems. Details are given in advanced text and reference books such as KJ Binns and PJ Lawrenson (1973) "Analysis and Computation of Electric and Magnetic Fields Problems" 2nd edn, Pergamon Press.