Aims and Objectives for Relativity 1 and Vectors: Session 17

SPACE-TIME EVENTS AND THE LIGHT CONE

Aims (What I intend to do)

1) To introduce the use of space-time diagrams.
2) To show how events in two different inertial reference frames , one moving with respect to the other, can be represented on the space-time diagram.
3) To look at simultaneity and length contraction with the aid of space-time diagrams.
4) To look at the role played by the space-time interval.

Objectives (What you should be able to do after completing the lecture and worksheet)

1) To be able to draw a space-time diagram, to show why light takes a slope of 45^o and to explain what is meant by the world line of a particle.

2) To be able to explain the reasoning for assigning different parts of the space-time diagram as Future, Past and Elsewhere.

3) To be able to show that the space-time interval

$$\Delta S^2$$

   is an invariant quantity under Lorentz transformations.
4) To be able to use a space-time diagram to help explain length contraction and simultaneity in Special Relativity.

Relativity 1 and Vectors: Worksheet 17

SPACE-TIME EVENTS AND THE LIGHT CONE

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1) Go over your lecture notes.

2) Take the expression,

$$c^2t'^2 - x'^2 - y'^2 - z'^2$$

   and use the Lorentz transformations to show that indeed it is an invariant quantity, i.e. that it equals,

$$c^2t^2 - x^2 - y^2 - z^2$$

3) Check that you can meet the objectives for this session.

Solution to task 2 of worksheet 16.

We can look at the invariance of energy and momentum conservation under Galilean transformations in the following way. We consider a perfectly elastic collision between two particles of mass *m* and *M*. In frame *S*, before the collision particle *m* has speed v_b and particle M has speed V_b. In frame S', a frame that moves w.r.t frame S at speed *u* in the x-direction, *m* has speed v_b' and M has speed V_b'. After the collision, in *S*, *m* has speed v_a and M has speed V_a whilst in S', *m* has speed v_a' and M has speed V_a'.

a) conservation of momentum: *momentum before = momentum after*

In S' this is,

$$m v_b' + M V_b' = m v_a' + M V_a' \tag{1}$$

Next we make use of the Galilean velocity transformation,

$$v' = v - u \tag{2}$$

to see how the momentum conservation equation is perceived in *S*.

$$m(v_b - u) + M(V_b - u) = m(v_a - u) + M(V_a - u)$$

which, cancelling terms, gives,

$$m v_b + M V_b = m v_a + M V_a \tag{3}$$

which is just the same as equation (1), but without the primes, conservation of momentum still holds.

a) conservation of energy: ***KE before collision =KE after collision***

In S' we have,

$$\frac{1}{2}m(v_b')^2 + \frac{1}{2}m(V_b')^2 = \frac{1}{2}m(v_a')^2 + \frac{1}{2}m(V_a')^2 \tag{4}$$

To find the viewpoint of an observer in *S*, we need to apply the Galiliean velocity transformation (equation (2) above), (4) then becomes,

$$\begin{aligned}&\frac{1}{2}m(v_b)^2 - mv_b u + \frac{1}{2}mu^2 + \frac{1}{2}M(V_b)^2 - MV_b u + \frac{1}{2}Mu^2\\ &= \frac{1}{2}m(v_a)^2 - mv_a u + \frac{1}{2}mu^2 + \frac{1}{2}M(V_a)^2 - MV_a u + \frac{1}{2}Mu^2\end{aligned} \tag{5}$$

Now, we have already shown that the conservation of momentum holds. As a result, all teerms involving *u* in equation (5) cancel. Further, all terms involving u^2 also cancel. Thus,

$$\frac{1}{2}m(v_b)^2 + \frac{1}{2}M(V_b)^2 = \frac{1}{2}m(v_a)^2 + \frac{1}{2}M(V_a)^2 \tag{6}$$

i.e. KE is conserved as viewed in *S*.

Solution to task 3 of worksheet 16.

Equation 8 gives, $\mathrm{x}' = \gamma(\mathrm{x\text{-}ut})$, and equation 9 gives, $\mathrm{x}' = \frac{\mathrm{x}}{\gamma} - \mathrm{ut}'$

equating these expressions we find,

$$\gamma(\mathrm{x\text{-}ut}) = \frac{\mathrm{x}}{\gamma} - \mathrm{ut}'$$

solving this for t',

$$\mathrm{ut}' = \frac{\mathrm{x}}{\gamma} + \gamma\mathrm{ut}\text{-}\gamma\mathrm{x}$$

so that

$$\begin{aligned}\mathrm{t}' &= \frac{\mathrm{x}}{\mathrm{u}\gamma} + \gamma\mathrm{t}\text{-}\frac{\gamma\mathrm{x}}{\mathrm{u}}\\ &= \gamma\mathrm{t} + \frac{\mathrm{x}}{\mathrm{u}}\left(\frac{1}{\gamma} - \gamma\right)\\ &= \gamma\mathrm{t} + \gamma\frac{\mathrm{x}}{\mathrm{u}}\left(\frac{1}{\gamma^2} - 1\right)\end{aligned}$$

now,

$$\frac{1}{\gamma^2} - 1 = 1 - \frac{u^2}{c^2} - 1 = -\frac{u^2}{c^2}$$

so that,

$$t' = \gamma t + \gamma \frac{x}{u} . \frac{-u^2}{c^2}$$

so that finally,

$$t' = \gamma \left(t - \frac{ux}{c^2} \right)$$

as required.