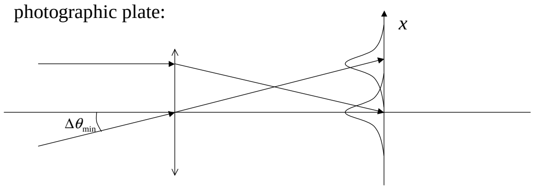
PHY2208 Lecture 15

Calculate the **diffraction-limited resolution** of an optical instrument.

Y&F section 38-8
Pedrotti & Pedrotti 16-3 & 16-4

Diffraction places a **fundamental limit** on the 'spatial resolution' achievable using an optical imaging system.

Spatial resolution is analogous to spectral resolution. Consider a lens forming an image to two distance point sources on a photographic plate:



The uncertainty principle suggests that each 'image' suffers an unavoidable blurring. Hence there exists a minimum angular separation $\Delta\theta_{\min}$ below which the images cannot be resolved.

For simplicity we can consider a 1-d lens, whose aperture transmission function is thus a slit of width a.

A single remote point source thus illuminates the lens with plane wavefronts, which are diffracted by the effective 'aperture' of width a.

The complex amplitude in the plane containing the geometric focus is then

$$\Psi(u) \propto \int_{-a/2}^{+a/2} \exp(-2\pi i u\xi)\, d\xi$$

where $u = x/(\lambda f)$ and f is the focal length of the 1-d lens.

We can evaluate this:

$$\Psi(u) \propto \left[\frac{\exp(-2\pi i u\xi)}{-2\pi i u}\right]_{-a/2}^{+a/2}$$

$$\Psi(u) \propto \frac{\sin(\pi u a)}{\pi u}$$

$$\Psi(u) \propto a\,\frac{\sin(\pi u a)}{\pi u a}$$

We define the **sinc** function: $$\frac{\sin(\pi x)}{\pi x} \equiv \mathrm{sinc}(x)$$

Note that there are **two** definitions of the sinc function in common use:

$$\frac{\sin(\pi x)}{\pi x} \equiv \mathrm{sinc}(x)$$

This definition is used in this course and has zeros at 0 and $x=n$ where n is an integer.

It has the convenient normalization $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\, dx = 1$

The alternative definition, which has zeros at 0 and $n\pi$ is:

$$\frac{\sin(x)}{x} \equiv \mathrm{sinc}(x) \quad \text{so} \quad \int_{-\infty}^{\infty} \mathrm{sinc}(x)\, dx = \pi$$

Two point sources produce two overlapping sinc functions.

Rayleigh criterion: These two patterns become indistinguishable when the primary maximum of one just overlaps the 1st minimum of the other i.e. we require a minium image separation:

$$\Delta x_{\min} = \frac{\lambda f}{a}$$

i.e. a minimum angular separation:

$$\Delta\theta_{\min} = \frac{\lambda}{a}$$

This is a very important result. Compare with the formula crudely derived from the Uncertainty Principle (Lecture 13).

Hence the image formed by our 1-d lens of width *a* of a point source at (image distance) infinity is described by:

$$I(x) \propto a^2 \, \text{sinc}^2\left(\frac{xa}{\lambda f}\right)$$

sinc(z) equals 1.0 at z = 0.0 and has zeros at $z = m$ with m any non-zero integer.

The (more realistic) 2-d rectangular aperture (dimensions $a \times b$) produces the pattern:

$$I(u,v) \propto \text{sinc}^2(ua) \cdot \text{sinc}^2(vb)$$

A 2-d circular aperture produces the pattern:

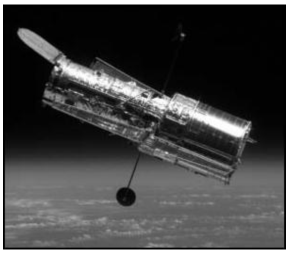
$$I(q) \propto \left(\frac{J_1(\pi q a)}{\pi q a}\right)^2 = \text{jinc}^2(qa)$$

where J_1 is the 1st order Bessel function of the first kind and q = $r/(\lambda f)$ with r being the radial distance from the centre of the pattern. jinc(z) has its first zero at z = 1.22 so:

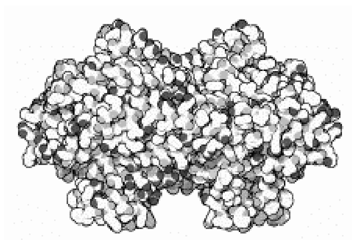
$$\Delta\theta_{\text{min}} = \frac{1.22\lambda}{a}$$

for a circular aperture of diameter *a*.

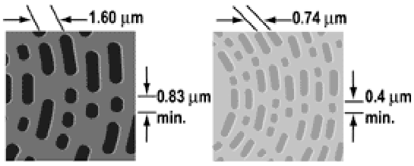
NB to increase the diffraction-limited resolution we must either increase *a* or decrease λ i.e. we want to decrease $\Delta\theta_{\text{min}}$



The Hubble Space Telescope (*a* = 2.5 m) has 500 times greater resolution than the human eye (a ~ 5 mm).



We use x-rays (λ ~ 1nm) in preference to optical radiation (λ ~ 500nm) to probe crystal and molecular structure.



DVDs use blue laser diodes (λ ~ 400nm) rather than red laser diodes (λ ~ 800nm) as they produces a smaller focused spot and allow a higher information density.

Now try problems 7 & 8 in Problem Sheet 2