General relativity and cosmology

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Matt Roos, Introduction to Cosmology, Wiley

IR Kenyon, General Relativity, Oxford Science

R D'Invernu, Introducing Einstein's Relativity, Clarenden Press

Syllabus

- Review special relativity: events; metric tensor; linear and non-linear transformations

- Assumptions leading to Lorentz transformation

- Examples of transformations

- Doppler effect

- Basic concepts of General relativity: non-inertial forces, equivalence principle, non-linear transformations, curved geometrical spaces

- Basic definitions of curvature, relationship with tidal forces;

- Einstein's field equations; motion of particle and photon in gravitational field;

- Schwarzschild metric; Maybe: orbits of particles; precession of perihelion of mercury; deflection of light by gravitational field;

- Black holes; gravitational red-shift;

  Maybe Finkelstein-Eddington coordinates; Kruskal-Szekeres coordinates;

- Cosmology: Roberson-Walker metric red-shift distance relation;

- Friedmann models; cosmic microwave background;

- Early Universe; helium production; Big-bang theory

These notes are intended to emphasise the principal results and provide questions which test understanding. They are supplemented by the lectures and text books.

Special relativity

Relativity is concerned with relationship between events observed in different reference frames.

Example: star exploding at a time and location as seen by two observers who are moving relative to each other.

A reference frame S is a particular frame adopted by an observer who can measure the spatial and temporal coordinates of an event. The time of the event is that recorded on a clock at rest in the frame.

Other reference frames may be moving either uniformly or non-uniformly (ie accelerating) wrt S.

Event is defined by time at which it occurs and location in space, eg (ct, x, y, z).

We are often concerned in GR with events that are close-by, eg change in location of particle in time dt. Denote such events by (ct, x) and $(ct + cdt, x + dx)$.

Events do **NOT** have the same coordinates in all reference frames.

If $x(p)$ is position of particle at time $t(p)$, then world line is path of particle in space-time, ie $(ct(p), x(p))$, as p changes.

Example: SHM $x = a\cos(p), \quad ct = cp/\omega$ then eliminating p gives $x = a\cos(\omega t)$.

Problem: Describe the world line of a particle executing 2-dimensional SHM

Problem: Write down the coordinates of two (different) examples of events.

Inertial reference frames

An inertial reference frame (IF) is one in which free particles move in straight lines with constant speed, ie $x = a + bt$.

Example: a horizontal frictionless table.

Example: frame fixed wrt to stars

Problem: Show that if a frame S' moves uniformly wrt to an IF, then free particles will also move in straight lines in S' and hence frame is inertial.

Problem: Show that if Newton's second law $\dot{p} = F$ where $p = m\dot{x}$ is valid in one IF S, then the same equation is valid in all frames moving uniformly wrt S and hence all IFs.

Laws of SR are valid in inertial reference frame. This means, that they need NOT be obeyed in non-inertial (eg accelerating) frames.

Problem: If S' accelerates uniformly wrt S, find and solve the equation of motion in S' of a particle which moves freely in S.

How does this differ from a particle subject to a constant force moving in S?

Space-time interval

Define space-time interval ds between two close-by events in an inertial reference frame by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

Example: The interval between ticks of a stationary clock at origin. Event corresponding to n-th tick is $c(ndt, 0, 0, 0)$ and interval is $ds^2 = c^2 dt^2$.

In this case, ds/c is called proper time interval.

Example: The length of a short rod whose ends are measured simultaneously. Events corresponding to the simultaneous observation of their ends are (ct, x) and $(ct, x + dx)$. Hence space time interval is given by: $ds^2 = -dx^2$.

In this case $\sqrt{-ds^2}$ is called proper length.

Proper time

Definition: If (ct, x) and $(ct + cdt, x + dx)$ are two events observed in an inertial reference frame, then proper time interval between them is $d\tau$ where $d\tau^2 = ds^2/c^2 = dt^2 - dx^2/c^2$.

Problems:

A particle moving with speed w along the x-axis. Find the proper time interval between two events on its world line (ct, x) and $(ct + cdt, x + dx)$ in terms of w and dt.

What is $d\tau$ if the particle is a photon?

What is $d\tau$ for interval between ticks of a clock moving along x axis with speed v?

Under what circumstance is $d\tau$ imaginary? Can this then be measured?

Transformation of axes

Show that: if $x - y$ axes rotated by α, in an anti-clockwise manner about z, then coordinates between two close-by points (originally (dx, dy)) are

$$dx' = dx \cos\alpha + dy \sin\alpha$$
$$dy' = dy \cos\alpha - dx \sin\alpha,$$

and that the reverse transformation is

$$dx = dx' \cos\alpha - dy' \sin\alpha$$
$$dy = dy' \cos\alpha + dx' \sin\alpha$$

Important point is ${dx'}^2 + {dy'}^2 = dx^2 + dy^2$ and hence in prime frame,

$$ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

is exactly the same as ds^2 even though coordinates of events differ.

The principle of relativity shows that this is also case for frames in uniform relative motion.

L2: Assumptions in special relativity

Aim: To develop Lorentz transformation giving relationship between coordinates of an event when viewed in two frames in uniform relative motion enumerate

Assumptions:

(1) Speed of light is independent of inertial reference frame;

(2) Free particles always move with constant speed on straight lines;

(3) Empty space is homogeneous and isotropic.

Implications:

(a) For any close-by events $ds'^2 = ds^2$ and these vanish for events on world lines of photons.

(b) Second assumption implies transformation between coordinates of an event on world line of free particle when viewed in two inertial reference frames S and S' is linear.

This also implies that transformation between S and S' is linear for any event.

(c) For prime frame, S', moving along x-axis with speed v, and which coincide at $t = 0$, then the relationship between any event x, t and the same event x', t' viewed in S' is given by the Lorentz transformation:

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx/c^2)$$
$$y' = y, \quad z' = z$$
$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

The Lorentz transformation

Derived for case where frame S' moves along x-axis withs speed v relative to S and frames coincide at origin at time 0.

For a light signal moving along the positive x axis, we have $x - ct = 0$ in S, and hence $x' - ct' = 0$ in S'. Now for arbitrary events, it follows from fact that transformation is linear, that

$$x' - ct' = A(x - ct) + D(x + ct)$$

where A, D depend only on speed of the S' frame. But LHS vanishes for light signal moving along x axis, and hence $D = 0$.

In a similar way, for light wave moving along $-x$, $x' + ct' = x + ct = 0$ and for arbitrary events

$$x' + ct' = B(x + ct)$$

where B just depends on relative speed of frames v.

First is derived for light moving in direction of v and second for light moving in direction $-v$.

Now consider the observation of events from the view point of S', who sees S moving along $-x$ with speed v. Then $x + ct = C(x' + ct')$. However, since space is isotropic, this is same transformation as when S' moves along x with speed v. Hence C must be the same as A.

Thus $(x + ct) = A(x' + ct') = AB(x + ct)$ and $AB = 1$.

Problem: Show as a consequence that ${x'}^2 - c^2{t'}^2 = x^2 - c^2t^2$, and hence space-time interval is an invariant:

Now to get A and B, consider location of origin of S' after time t. In frame S this is (ct, vt) and in frame S' is $(ct', 0)$.

$$-ct' = A(v - c)t$$
$$ct' = B(v + c)t$$

Hence $-A(v-c) = B(v+c) = (v+c)/A$ or $A = \gamma(1+v/c)$ and $B = \gamma(1-v/c)$ remembering $AB = 1$. Here $\gamma = 1/\sqrt{(1-v^2/c^2)}$.

Finally, Lorentz transformation is

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - (v/c^2)x)$$

Problem: If $(ct, x) = (2, 3)$, find (ct', x') for S' moving along x with speed $c/2$. Verify space time interval is invariant.

Problem: Prove that the Lorentz transformation can be written in the form

$$x' = x\cosh\phi - ct\sinh\phi$$
$$ct' = -x\sinh\phi + ct\cosh\phi$$

and identify ϕ.

Problem: Write down the Lorentz transformation for observer moving along (a) $-x$ axis with speed v, (b) y-axis with speed v.

Problem: An observer stands midway between two light bulbs situated at $-l$ and l along the x-axis and which are switched on at $t = 0$. A second observer moves along the line joining the bulbs with speed v and is midway between the bulbs at $t = 0$. Find the times when each observer detects light from each bulb.

L3: Transformation of velocity

Consider a particle moving with velocity $(w_x, w_y, 0)$ in frame S. We require velocity in frame S'.

To obtain this, we use the Lorentz transformation,

$$dx' = \gamma(dx - vdt)$$
$$dt' = \gamma(dt - (v/c^2)dx)$$

If $dx = w_x dt$ and $dx' = w'_x dt'$ then

$$w'_x = \frac{dx'}{dt'} = \frac{(w_x - v)}{1 - vw_x/c^2}$$

Problem show:

$$w'_y = \frac{w_y}{\gamma(1 - vw_x/c^2)}$$

Problem: A car moves along x axis with speed $3c/4$ and meets a car travelling in opposite direction with speed $3c/4$. What is relative speed of the cars?

Stellar aberration

Light from a star is seen to arrive at a fixed origin in frame S at an angle α with x-axis. What is angle in the frame S'?

Velocity of light moving towards origin in S is $\mathbf{w} = (-c\cos\alpha, -c\sin\alpha, 0)$. In S',

$$w'_x = -\frac{(c\ \cos\ \alpha + v)}{1 + v/c\ \cos\ \alpha)}$$

$$w'_y = -\frac{c\ \sin\ \alpha}{\gamma(1 + v/c\ \cos\ \alpha)}$$

$$w'_z = 0$$

Problem: Show speed of light in frame S' is c.

Velocity in S' is $c(-\cos\ \alpha', -\sin\ \alpha', 0)$

where

$$\tan\ \alpha' = \frac{\sin\ \alpha}{\gamma(\cos\ \alpha + v/c)}$$

For ultra-relativistic case, v tends to c, and use $\sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$, and $1 + \cos\alpha = 2\cos^2(\alpha/2)$ to get

$$\tan\alpha' \sim \frac{1}{\gamma}\tan\alpha/2$$

Problem: show this $\rightarrow 0$ as v tends to c. Hence no starlight reaches observer except that coming in at very low angles α. Implies night sky is highly anisotropic to moving observer.

Problem:

A star is seen to lie at an elevation of 45°. What are elevations for observer moving at $0.5c$, $0.9c$ and $0.99c$?

Space - time cone

Distinguish 3 types of events:

a) time-like $ds^2 > 0$

b) space-like $ds^2 < 0$

c) light like $ds^2 = 0$.

If the space-time interval between two close-by events is space-like, then there exists a reference frame in which the relative coordinates can be written as

$$(0, dx, 0, 0).$$

Note space-like means $ds^2 < 0$ and $dl = \sqrt{\{dx^2 + dy^2 + dz^2\}} > cdt$. First orientate the frame so that $dy = dz = 0$ and cdt/dx must be less than 1. Transform to a new frame so that

$$cdt' = \gamma(cdt - vdx/c)$$
$$dx' = \gamma(dx - vdt)$$

Choose the frame where $cdt' = 0$ and this implies $cdt - vdx/c = 0$. Thus need to select v so that $v/c = cdt/dx$. This is always possible as RHS has magnitude less than 1.

Problem: if the interval is time-like, show there exists a frame for which

$$(cdt, 0, 0, 0)$$

Relativistic Doppler effect

Aims: To deduce frequency shift in light waves caused by relative motion of source and observer

For source at origin of S' emitting light at time t which is detected by S at time T. Then

(1) for light emitted along along x-axis,

$$\begin{aligned} c(T-t) &= x \\ c(dT-dt) &= dx = vdt \\ dT &= (1+v/c)dt \end{aligned}$$

But interval dt in S corresponds with dt' in S' where $dx' = 0$ as source is at origin in S'. Hence from the LT, $dx' = 0$ implies $dx = vdt$ and $dt' = \gamma(dt - vdx/c^2) = dt/\gamma$. Thus $dT = \gamma(1+v/c)dt'$.

Now the frequency in frame S is related to $1/dT$ and to proper frequency observed in S' (which is $1/dt'$) by $\nu_0/\nu = (1+v/c)\gamma$ or $\lambda/\lambda_0 = (1+v/c)\gamma$.

and $\lambda > \lambda_0$ (red-shift). This is longitudinal Doppler effect.

(2) for case where where light is emitted along y axis,

$$c(T - t) = y$$
$$c(dT - dt) = 0$$
$$\text{or } dT = dt$$

Hence as dt in S corresponds with $dt' = dt/\gamma$. Hence $dT = \gamma dt'$ and $\lambda = \lambda_0 \gamma$. Effect is very much weaker and second order in v/c. This is transverse Doppler effect.

Problem: Evaluate the red-shift $\Delta\lambda/\lambda$ at the origin for a light signal on on the rim of a centrifuge of radius 0.9311 m and spinning at 43000 rpm.

Ans 9.76×10^{-13}

Doppler effect II

Previous theory was valid for the source S' moving away from observer S. Consider now case when S' is behind S so that light moves along positive x axis towards S.

If light emitted by S' at t (negative) and reaches S at T, then $T - t = x = -vt$. The sign in last term comes from fact that $T - t > 0$.

Problem: Work this through to give

$$dT = (1 - v/c)dt = \gamma(1 - v/c)dt'$$

$$\lambda/\lambda_0 = (1 - v/c)\gamma = \sqrt{(1 - v/c)/(1 + v/c)}$$

This is blue shift.

Metric tensor

Aims: To introduce metric tensors η and g;

Alternative way of writing ds^2

$$ds^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{00} = 1, \quad \eta_{11} = -1, \quad \eta_{22} = -1, \quad \eta_{33} = -1 \quad \eta_{\mu\nu} = 0 \quad \mu \neq \nu$$

$\eta_{\mu\nu}$ is called the metric tensor for flat space-time.

Very often in GR, the summation over an equation containing a lower and upper index like μ, ν is implied and hence above is written

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

Problem: Write out the full equation for ds^2.

Metric tensors were first invented to describe distances between two points on curved surfaces, or in flat spaces with curvilinear coordinates. For example, in Cartesian, spherical, cylindrical coordinates:

$$dl^2 = dx^2 + dy^2 + dz^2$$
$$dl^2 = dr^2 + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2$$
$$dl^2 = dz^2 + \rho^2 d\theta^2 + d\rho^2$$

In each case the distance between two close-by points can be written:

$$dl^2 = g_{\mu\nu}(x^1, x^2, x^3) dx^\mu dx^\nu$$

where x^1, x^2, x^3 are r, θ, ϕ etc.

We shall introduce later two different metric tensors appropriate for case when gravitational field present.

Schwarzschild metric in polar coordinates

$$ds^2 = c^2 d\tau^2 = B(r)c^2 dt^2 - \frac{1}{B(r)} dr^2 - r^2 d\phi^2,$$

$$B(r) = 1 - \frac{2GM}{c^2 r}$$

Here M is the point mass situated at the origin.

Robertson-Walker metric

$$ds^2 = c^2 dt^2 - R^2(t)\{\frac{1}{1-kr^2}dr^2$$
$$+r^2 \sin^2\theta \; d\phi^2 + r^2 d\theta^2\}$$

These metrics differ from ones in special relativity because frames are accelerating wrt an IF and there is gravitational field present.

Problem:

What is proper distance between two points r, θ, ϕ and $r + dr, \theta, \phi$ in Schwarzschild metric.

Problem:

The space-time coordinates (ct', x') of an event in a frame of reference S which is accelerating with respect to a local inertial reference frame S are:

$$x' = e^x, \quad ct' = ct + x,$$

Here (ct, x) are the coordinates of the same event in the local inertial frame where $ds^2 = c^2dt^2 - dx^2$.

(a) Find ct and x in terms of ct' and x'.

(b) Find cdt and dx in terms of ct' and x'?

(c) Hence find the metric in the accelerating frame.

Metric tells us 'distance' between events in space time. $d\tau = ds/c$ is still proper time or time measured by a stationary clock $(dx = dy = dz = 0)$ but this depends on location.

For Schwarzschild metric, $dt = d\tau/(1-2MG/rc^2)$ or $dt > d\tau$ and hence clock near Sun slows down wrt a clock measuring proper time.

Photons still move so that $ds = 0$.

Proper distance is still related to $\sqrt{-ds^2}$ and would be spatial distance between two simultaneous events.

The question then arises as to what determines the metric?

General Relativity

Aim: To introduce equivalence principle where gravitational field is identified with an accelerating reference frame

Recall Newton's Gravitation law

$$F = m_g g, \quad g = -GM/r^2, \quad F = m_i \ddot{x}$$

m_g and m_i are called gravitational and inertial masses. Precise experiments show these are equal but there is no explanation for this in classical gravitational theory.

Forces which induce the same acceleration in particles irrespective of the mass of the particle upon which they act are called inertial. If $m_i = m_g$, then gravitation is an inertial force

If particle is moving relative to a lift accelerating vertically upwards with acceleration a, then

$$m_i \ddot{y} = -m_g g - m_i a$$

where y is local coordinate, $ie.$ coordinate relative to lift floor.

Note that gravitational force can be eliminated if choose frame in free fall where $a = -m_g g/m_i$ which is independent of mass if $m_i = m_g$. Thus inertial forces can be eliminated by change of reference frame.

This demonstrates that an inertial force is introduced when a frame accelerates wrt an inertial one. Note that observer S' seeing an accelerating particle, which is free in frame S, would not be able to distinguish whether S' is accelerating wrt S or whether there is inertial force acting on particle.

Thus equivalence principle states that gravity is an inertial force so that $m_i = m_g$ and gravity can be eliminated locally by choice of frame in free fall also called the local inertial frame (LIF). In addition, the laws of special relativity are obeyed locally in LIF. Important to note tidal forces are not eliminated.

Equivalence Principle

Consequences of principle:

a) Bending of light by gravitational field;

If Sun exert gravitational force downwards, then equivalent to upward accelerating lift as equation of motion of particle in gravitational field $m\ddot{y} = -mg$ is same as $m\ddot{y} = -m\ddot{L}$ where y is height above lift floor and L is height of lift above ground (LIF).

A light ray in upward accelerating lift moving horizontally in a straight line, strikes the sides of the lift. So observer sees it striking sides of lift at heights y_1 and y_2 where $y_2 = y_1 - \ddot{L}t^2/2$ where t is time taken for light to cross lift. But if this frame is equivalent to presence of downward gravitational field, then it appears to the stationary observer, that light is bent towards the source of gravity.

b) Gravitational red-shift.

Consider light moving vertically upwards away from a gravitational source like the Sun towards the Earth. If the observer on surface of Sun is replaced by one in a lift in downward (towards Sun) motion, ie in free fall, then this frame appears to be moving away from Earth. Hence light detected in this frame due to atomic transition is then Doppler red shifted with respect to same atomic transitions on Earth.

Curvature

Aim: To introduce extrinsic, intrinsic, and Gaussian curvature of a surface

Curvature is central to the theory of relativity. There are two versions in the geometry of curved surfaces: extrinsic and intrinsic.

Curvature K of a simple curve in 2-dimensions is defined as $\frac{\partial \psi}{\partial s}$ or rate of change of angle made by tangent at point P on curve and x-axis, with respect to arc length.

Introduce a circle drawn through P with radius R (radius of curvature) $R = 1/K = \frac{\partial s}{\partial \psi}$. Then arc length ds along curve equals arc length along circle, ie circle osculates with curve at P.

Problem: Show for a curve $y = y(x)$,

$$K = \frac{\frac{d^2y}{dx^2}}{\left(1 + (\frac{dy}{dx})^2\right)^{3/2}}$$

Problem: Evaluate curvature of $y = x^2$ and $y = -x^2$.

Extrinsic curvature of surface

Choose $x - y$ axes lying in tangent place to surface at P. Choose a line L in the $x{-}y$ plane making an angle θ with the x-axis. Surface projected onto $z - L$ plane makes a curve with curvature $K(\theta)$ at P. Principal values of curvature of surface at P are maximum, K_1, and minimum, K_2, values of $K(\theta)$.

Gaussian curvature

Define Gaussian or intrinsic curvature of surface by $K_1 K_2$.

Problem: Show Gaussian curvature of plane and cylinder are 0 while that of a sphere is $\frac{1}{r^2}$

Curvature of Torus having mean diameter d and cross sectional radius r.

Curvature at outermost point from centre is $(\frac{1}{r}) \times (\frac{1}{d/2+r})$, while on the inner surface at distance $r - d/2$ from centre, we have a negative curvature $(\frac{-1}{r}) \times (\frac{1}{d/2-r})$.

Important property of Gaussian curvature is that it is determined completely by metric.

Proof: First look at case where

$$ds^2 = g(r)dr^2 + r^2 d\theta^2$$

Example: Consider sphere, with radius a, resting on horizontal surface at point Q. Choose coordinate system with origin at Q. Coordinates of a point on sphere are (x, y, z). Then in cylindrical coordinates r, θ, z $x = r\cos\theta$, $y = r\sin\theta$ with $x^2+y^2+(z-a)^2 = a^2$.

Distance between two points on sphere $ds^2 = dx^2 + dy^2 + dz^2$ with $(z-a)^2 = a^2 - x^2 - y^2$. Now use

$$
\begin{aligned}
dx &= dr\cos\theta - r\sin\theta d\theta \\
dy &= dr\sin\theta + r\cos\theta d\theta \\
z &= a \pm \sqrt{a^2 - r^2} \\
dz &= -\frac{rdr}{\sqrt{a^2 - r^2}} \\
ds^2 &= dr^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{a^2 - r^2} \\
&= \frac{a^2 dr^2}{a^2 - r^2} + r^2 d\theta^2 \\
g &= \frac{a^2}{a^2 - r^2}
\end{aligned}
$$

Hence metric in these coordinates is of the form

$$ds^2 = g(r)dr^2 + r^2 d\theta^2$$

Now consider great circle passing through Q and a point P with coordinates r, θ, z lying on surface of sphere. We want the curvature at P. Curvature of great circle passing through P in $z - r$ plane is $K_1 = 1/a$.

Consider another circle centred on z-axis and passing through P. Radius is r and curvature K_2 is found from $K_2 = \sin\psi/r$ where ψ is angle between z-axis and normal to tangent plane at P. But $\sin\psi = r/a$ and thus $K_2 = 1/a$ or Gaussian curvature is $1/a^2$.

For more general case when $g(r)$ is an arbitrary function of r, $K_1 = \frac{\partial\psi}{\partial s}$ and $K_2 = \frac{\sin\psi}{r}$ where ψ is angle between the z-axis and the normal to surface at P. It is also equal to the angle between tangent at P and r axis . Hence, $\cos\ \psi = \frac{\partial r}{\partial s} = \frac{1}{\sqrt{g(r)}}$. Thus Gaussian curvature $K = K_1 K_2 = \frac{1}{2rg^2(r)}\frac{\partial g(r)}{\partial r}$.

This demonstrates that Gaussian curvature is completely determined by metric. This is a general result.

Problem: Show this formula gives curvature of sphere when we use $g(r) = \frac{a^2}{a^2 - r^2}$

Example: paraboloid of revolution

To find curvature of surface $z = ar^2$ at point P with cylindrical coordinates (r, θ, z). First note distance between two points on surface is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 = g(r)dr^2 + r^2 d\theta^2$$
$$g(r) = 1 + 4a^2r^2$$

The intersection of the surface with the $(r, \theta = 0, z)$ plane, results in a curve $z = ar^2$ whose curvature is found from that of a curve in 2-dimensions.

$$K_1 = \frac{\frac{d^2y}{dx^2}}{\left(1 + (\frac{dy}{dx})^2\right)^{3/2}}$$

$$= \frac{\frac{d^2z}{dr^2}}{\left(1 + (\frac{dz}{dr})^2\right)^{3/2}} = \frac{2a}{(1 + 4a^2r^2)^{3/2}}$$

To get second principal curvature, draw an osculating circle through P so that its radius lies along normal to P and whose tangent at P is perpendicular to the $(r, \theta = 0, z)$ plane. The radius of this circle is $r/\sin\psi$ where ψ is angle between z-axis and the normal to surface at P. Note

$$\sin\psi = \frac{dz}{ds} = \frac{\frac{dz}{dr}}{\frac{ds}{dr}}$$

$$= \frac{\frac{dz}{dr}}{\sqrt{(1 + (\frac{dz}{dr}^2)}} = \frac{2ar}{\sqrt{1 + 4a^2r^2}}$$

Hence curvature

$$K_2 = \frac{\sin\psi}{r} = \frac{2a}{\sqrt{1 + 4a^2r^2}}$$

and Gaussian curvature is

$$K = K_1 K_2 = \frac{4a^2}{(1 + 4a^2r^2)^2}$$

whereas Gaussian curvature is also given by

$$K = \frac{1}{2rg^2(r)} \frac{\partial g(r)}{\partial r} = \frac{8a^2r}{2r(1+4a^2r^2)^2} = \frac{4a^2}{(1+4a^2r^2)^2}$$

which is same!

Geodesics

Geodesics are paths taking the shortest distance between two points. These are straight lines in Euclidean geometry.

On sphere, the geodesics are great circles. If two of these intersect at angle ϕ at North Pole, then the angular distance between them as one moves a distance s from the pole along each geodesic is

$$\eta = R\phi \sin(\frac{s}{R})$$

which obeys the differential equation

$$\frac{d^2\eta}{ds^2} = -K\eta$$

Note this is similar to Newton's gravitational law where left hand side is acceleration and rhs is acceleration due to gravity. This establishes a link between gravity and curvature.

In SR, free particles and photons move along straight lines with constant speed. These are geodesics. In presence of gravitation, the particles still move along geodesics but these are no longer ‘straight’ lines as tidal forces now operate.

Basic principles of General Relativity

Aim: To introduce without detail the principles of General Relativity

The equations of motion of two close-by particles at x and $x + \delta x$ moving in a potential are

$$
\begin{aligned}
m\ddot{x} &= F(x) = -m\frac{d\Phi}{dx} \\
m(\ddot{x} + \ddot{\delta x}) &= F(x + \delta x) \\
m\delta\ddot{x} &= \frac{dF}{dx}\delta x \\
&= -m\frac{d^2\Phi}{dx^2}\delta x
\end{aligned}
$$

RHS is tidal force.

Problem: Explain why there are two daily tides.

One can apply the same reasoning and introduce a tidal force in relativity. There are three steps:

a) Transform equation of motion for free particle in LIF $\ddot{\zeta}^{\mu} = 0$ to accelerating reference frame x^{μ}.

b) Write down equation for relative motion of two close-by particles δx^{μ}.

c) Rewrite equation so that acceleration term is a tensor. Equation becomes

$$\frac{D^2 \delta x^{\mu}}{D\tau^2} = R^{\mu}_{\alpha\sigma\kappa} \delta x^{\sigma} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\kappa}}{d\tau}$$

Here R is Riemann curvature tensor and $\frac{DA^{\mu}}{D\tau}$ is called covariant derivative of A along the world-line. Analogous to equation for geodesic deviation

$$\frac{d^2\eta}{ds^2} = -K\eta$$

R is completely determined by metric tensor g but relationship is very complicated and can be found in text books.

Problem: The space-time coordinates x^{μ} of an event in a frame of reference S which is accelerating with respect to a local inertial reference frame are:

$$x^0 = \zeta^0, \quad x^1 = \exp \zeta^1, \quad x^2 = \zeta^2, \quad x^3 = \zeta^3.$$

Here ζ^μ are the coordinates of the event in the local inertial frame.

Write down the equation of motion of a free particle in the local inertial reference frame described by ζ^μ.

Find the derivatives $\partial\zeta^\mu/\partial x^\nu$ and hence find the equations of motion of the free particle in the accelerating frame.

Einstein's field equations

In limit when $c \longrightarrow \infty$, equations of relative motion of two particles given must above reduce to Newton's equations for tidal forces. These equations involve the gravitational potential which is related to the density of matter. Hence, the curvature itself is related to the matter density and this relationship is embodied in Einstein's field equations.

In Newtonian theory, gravitational acceleration is given by gravitational potential Φ obeying $\nabla^2\Phi = 4\pi G\rho$ (Poisson equation) with ρ being the density of matter. The equations for particle and relative acceleration of close-by particles (in one-dimension) are

$$\ddot{x} = -\frac{\partial \Phi}{\partial x}$$

$$\delta\ddot{x} = -\frac{\partial^2 \Phi}{\partial x \partial x}\delta x$$

Example of last equation is two daily tides caused by Moon where $\Phi = -GM/r$.

Problem: Show $\Phi = -r^2\omega^2/2$ at a point at a distance r from the axis of a frame rotating with angular velocity ω.

Hence in GR, tidal equation must reduce to Poisson's equation. Solution of Einstein's equations gives metric.

Now a consequence of Einstein equations is for static systems $ds^2 = c^2 d\tau^2 = g_{00}(x)c^2 dt^2 + g_{ij}dx^i dx^j$.

In limit $c \to \infty$, only time term important and then $g_{00}(x) = 1 + 2\Phi/c^2 = 1 - \frac{2GM}{c^2 r}$ for point mass.

This shows relationship between source of gravity and metric, and hence orbits of particles which move along geodesics.

A point mass M at $r = 0$ leads to the Schwarzschild metric

$$ds^2 = c^2 d\tau^2 = B(r)c^2 dt^2 - \frac{1}{B(r)}dr^2 - r^2 d\phi^2,$$

$$B(r) = 1 - \frac{2GM}{c^2 r}$$

The Schwarzschild metric

The Schwarzschild metric

$$ds^2 = c^2 d\tau^2 = B(r)c^2 dt^2 - \frac{1}{B(r)}dr^2 - r^2 d\phi^2,$$

$$B(r) = 1 - \frac{2GM}{c^2 r}$$

This shows that the coordinate time interval dt for a stationary clock $dr = d\phi = 0$ is much longer than the proper time $d\tau$. The proper distance between two events is given by $dl^2 = \frac{1}{B(r)}dr^2 + r^2 d\phi^2$.

For r small, proper length of a rod parallel to r is much longer than coordinate length, but lengths in transverse direction unaffected.

For $r < r_s$ terms change sign – means that length and time become interchanged and time coordinate is really r and space described by t.

This shows coordinate system (ct, r, ϕ) is unsatisfactory.

Newton's gravitation theory

To show planets move in eliptical orbits in Newtonian theory

Consider motion of planet of mass m in $x-y$ plane. The velocity of the planet has radial and tangential components equal to $\dot{r}$ and $r\dot{\phi}$. Since the total energy per unit mass and angular momentum per unit mass are constants, then

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} = constant$$

$$r^2\dot{\phi} = J'$$

Hence

$$\dot{r}^2 + \frac{J'^2}{r^2} - \frac{2GM}{r} = E'$$

Planet moves between values of r where

$$\frac{J'^2}{r^2} - \frac{2GM}{r} < E'$$

Problem: Plot left hand side as a function of r. Show that, in general, there are two values of r where left hand side equals right hand side.

Turning points satisfy quadratic equation for $u = 1/r$

$$J'^2 u^2 - 2GMu = E'$$

This yields in general two distinct values corresponding to aphelion and perihelion (greatest and smallest values of r). Let corresponding values of r be $1/u_1, 1/u_2$ respectively.

To get orbit, need to find ϕ as function of r. Substitute

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{J'}{r^2}\frac{dr}{d\phi}$$

Hence

$$\frac{J'^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{J'^2}{r^2} - \frac{2GM}{r} = E'$$

From $\frac{d\phi}{dr}$ we get

$$\phi = \int \frac{dr}{r^2} \frac{1}{\sqrt{2GM/(J'^2 r) - 1/r^2 + E'/J'^2}}$$

Introduce $V(r) = 2GM/(J'^2 r) - 1/r^2$, then

$$\phi = \int \frac{dr}{r^2} \frac{1}{\sqrt{E'/J'^2 + V(r)}}$$
$$= \int du \frac{1}{\sqrt{(u-u_1)(u_2-u)}}$$

where once again $u = 1/r$. The integral can be carried out exactly by substituting $u = \frac{u_1+u_2}{2} + \frac{(u_2-u_1)\cos\phi}{2}$ or $\frac{1}{r} = \frac{u_1+u_2}{2} + \frac{(u_2-u_1)\cos\phi}{2}$. This is equation of ellipse.

Note angle swept out when planet moves between aphelion and perihelion is π. Motion of planet is a periodic function of ϕ as increasing ϕ by 2π, yields same value of r. This is not true in GR.

Precession of Mercury

For a point mass M at the origin, metric in plane polar coordinates is due to Schwarzschild and takes the form:

$$ds^2 = B(r)c^2dt^2 - \frac{1}{B(r)}dr^2 - r^2d\phi^2, \; B(r) = 1 - \frac{2GM}{c^2r}$$

From the metric, the equations of geodesics, which are also the world lines of all particles, can be found. If world line is parametrised in terms of p, then geodesics, for planar orbits where $\theta = \pi/2$, are given by

$$\frac{dt}{dp} = \frac{1}{B}, \quad \frac{d\phi}{dp} = \frac{J}{r^2}, \quad \frac{d\tau}{dp} = \sqrt{E}$$

where E and J are constants while $E > 0$ for material particle and $E = 0$ for photon. First equation tells us time is distorted in gravitational field while second relates to conservation of angular momentum.

Third tells us world-line of material particle can be characterised by clock moving with particle, ie, p can be chosen related to τ.

Substituting $d\tau$ and dt in Schwarzschild metric gives along the geodesic:

$$Ec^2 = \frac{c^2}{B} - \frac{1}{B}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^2}$$

Express ϕ in terms of r by writing

$$\frac{dr}{dp} = \frac{dr}{d\phi}\frac{d\phi}{dp} = \frac{J}{r^2}\frac{dr}{d\phi}$$

gives

$$\frac{J^2}{Br^4}\left(\frac{dr}{d\phi}\right)^2 = \frac{c^2}{B} - \frac{J^2}{r^2} - Ec^2$$

Hence,

$$\phi = \int \frac{dr}{r^2\sqrt{B}} \frac{1}{\sqrt{c^2/(J^2B) - Ec^2/J^2 - 1/r^2}}$$
$$= \int \frac{dr}{r^2\sqrt{B}} \frac{1}{\sqrt{V + (1-E)c^2/J^2}}$$

where

$$V = c^2/(J^2B) - 1/r^2 - c^2/J^2$$

For small M, $1/B \sim 1 + 2MG/rc^2$ thus $V \sim 2MG/(J^2r) - 1/r^2$. This reduces to Newtonian case above where $J' = J$ and $E' = (1-E)c^2$.

The planet must move to ensure

$V(r) > (E-1)c^2/J^2$

The dependence of r on ϕ (eqn. of the orbit) is found by integration:

$$\phi = \int^r \frac{dr}{r^2\sqrt{B}} \frac{1}{\sqrt{c^2/(J^2 B) - Ec^2/J^2 - 1/r^2}}$$

However, difficult to carry out. Difference from Newtonian theory is no longer have quadratic equation for $1/r$ and although there are two turning points, r_1, r_2, corresponding to aphelion and perihelion, angle swept out when planet moves between them is not π but a value greater than π. For small $MG/(c^2 l)$, where

$$1/l = (1/r_1 + 1/r_2)/2$$

perihelion advances approximately at rate $6MG\pi/lc^2$ per revolution

$$\Delta\phi = \frac{6\pi GM}{c^2 \tau l}$$

For Mercury $MG/c^2 = 1.475$ km and $l = 55.3 \times 10^6$ km. Hence $\Delta\phi = 0.1038''$. Now Mercury makes

415 revolutions per (Earth) century. Hence

$$\Delta\phi = 43.08''/\text{century}$$

Observed value, after corrections, is $43.11''$/century

Observations go back to 1765. This is most important evidence of theory of GR. Many corrections to data required. Newtonian theory gives $5557''$/century due primarily to rotation of Earth based coordinate system. There is a contribution of $532''$/century due to gravitational fields of other planets. Other possibilities excluding GR for the effect are that the Sun is oblate. However, the latter would produce effects on other planets which are not observed.

Black holes

Consider motion of particles in Newtonian theory which are moving radially so that $J' = 0$. From orbital equations:

$$\dot{r}^2 - \frac{2GM}{r} = E'$$

E' is related to energy at $r = \infty$ and particle reaches there if

$$\dot{r} > \sqrt{(2GM/r)}$$

Rev. Mitchell (1784) considered this applied to photons and hence they could not escape gravitational field if

$$c < \sqrt{(2GM/r)}$$

Thus star would appear black.

Radius of star is then less than Schwarzschild radius

$$r_s = 2GM/c^2$$

This is same result as found in GR.

To take relativity into account, consider an observer or particle moving radially so that $J = 0$ and $E = 1$. Use geodesic equations

$$\frac{dt}{dp} = \frac{1}{B}, \quad \frac{d\phi}{dp} = \frac{J}{r^2}, \quad \frac{d\tau}{dp} = \sqrt{E}$$

$$Ec^2 = \frac{c^2}{B} - \frac{1}{B}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^2}$$

We find, as $E = 1$, $\tau = p$ and as $B = 1$ at infinity, then from $\frac{dt}{dp} = 1/B$, we get $t = p$. Substituting $E = 1$, gives $\frac{dr}{d\tau} = 0$ at infinity or observer starts from rest moving radially being pulled in by gravitational field. To find world line, note

$$c^2 = \frac{c^2}{B} - \frac{1}{B}\left(\frac{dr}{d\tau}\right)^2$$

$$\left(\frac{dr}{d\tau}\right)^2 - \frac{2GM}{r} = 0$$

For inward moving observer reaching $r = 0$ at τ_0

$$\frac{dr}{d\tau} = -(2GM/r)^{1/2}$$
$$\tau = \tau_0 - \frac{1}{3\sqrt{GM}} r^{3/2}$$

Thus to travel between two r values takes a finite time and one measured by co-moving clock with observer.

However, coordinate time taken to reach origin is singular!

$$\frac{dt}{d\tau} = \frac{1}{B}$$
$$d\tau = Bdt = -dr(\frac{r}{2GM})^{1/2}$$

$$dt = -dr \frac{r^{3/2}}{(2GM)^{1/2}(r - r_s)}$$
$$t \sim \ln(r - r_s)$$

Clearly this diverges as $r \to r_s$. This would be measured by stationary clock on Earth at a great distance from co-moving clock. Thus age of an observer on Earth greatly exceeds that of a moving observer when latter approaches r_s.

Although it is true that coordinate time taken for astronaut to reach r_s as measured by Earth based observer, greatly exceeds proper time, the coordinate time taken for light to travel back from astronaut to Earth also diverges. This follows from

$$Bc^2 dt^2 = dr^2/B$$
$$\frac{dr}{dt} = -Bc = -(1 - r_s/r)c$$
$$\frac{dt}{dr} = -\frac{r}{c(r - r_s)}$$

Nevertheless the singularity at $r = r_s$ is not a physical one. There is no mass at $r = r_s$ and curvature is not singular.

Singularity at $r = r_s$ is example of coordinate singularity. eg: introduce new distance coordinate for point on x-axis by $\eta = \tan x$. This is infinite at $x = \pi/2$ but singularity is entirely due to choice of coordinate.

Gravitational Red Shift

To derive equation for Gravitational red shift

The gravitational red shift can be understood from the equivalence principle. Two identical clocks have the same proper time, ie the coordinate time interval between ticks in a local inertial reference frame where gravity is absent and the clocks are at rest. In an accelerating frame, the proper time interval is unchanged but the coordinate time interval now is different from its value in the LIF. According to the EP, the coordinate time interval in the accelerating frame is the same interval observed when the clocks lie in a gravitational field. Thus the coordinate time intervals of the two clocks reflect the local gravitational fields but the proper time intervals between ticks is the same for both clocks. If the clocks are represented by atomic transitions on the Sun and Earth, then the coordinate time intervals are

$$dt_s = d\tau/\sqrt{g_{00}(s)}, \quad dt_e = d\tau/\sqrt{g_{00}(e)}.$$

As $g_{00} = 1 - 2MG/rc^2$, then $dt_s > dt_e$. Now light corresponding to one tick emitted by Sun travels through space to the Earth along a null-geodesic taking say 8 minutes. A second tick made by the clock on the Sun after dt_s would reach Earth at 8 minutes plus dt_s. Hence the clock on Sun is observed to have a wave-length proportional to dt_s which is longer that the wavelength of the Earth based clock proportional to dt_e. Thus the red shift

$$z = \frac{\lambda_s - \lambda_e}{\lambda_e} = \frac{\nu_e - \nu_s}{\nu_s} = \sqrt{g_{00}(e)/g_{00}(s)} - 1 \sim \frac{MG}{c^2 r_s}$$

For Sun, $r_s = 6.9 \times 10^5 \; km$, hence red-shift is 2.1×10^{-6}.

Problem is Döppler shift of light emitted by Sun caused by thermal currents. Measurement give red-shift to be 1.05 ± 0.05 times theoretical value. Earth based Mossbauer measurement: give red shift $(2.57 \pm 0.26) \times 10^{-15}$ compared with theoretical value of 2.46×10^{-15}.

Cosmology: standard coordinates

Aim: to introduce Cosmological Principle and cosmic standard coordinates

Cosmological principle: Universe is spatially homogeneous and isotropic on a sufficiently large scale:

diameter of galaxy 10^5 lt. yrs.

separation $\approx 10^6$ lt yrs. Scale for CP: 10^8 lt. yrs.

CP not true for all observers: why?

Cosmic standard coordinates:

Place clock at every galaxy; measure time on local clock; divide up space with set of grid lines tied to galaxies: galaxies labelled by intersection of grid lines. Note as time evolves, grid lines move with galaxies and hence spatial coordinates of galaxy x^i are constant.

Assumes galaxies do not collide!

Time for any event is measured by local clock $\rightarrow dx^i = 0, d\tau = dt$.

In absence of rotation, no terms in $dtdx^i$ because of time reversal symmetry:

(All fundamental laws we know are invariant under time reversal- friction and irreversibility consequence of statistical mechanics)

Measure proper distance to two stars , one along say x-axis and the other along another axis, say y, from observer at origin.

$$dl_1^2 = -g(t,0)dx_1^2$$
$$dl_2^2 = -g(t,0)dy_2^2$$

At later time $t = t'$, re-measure:

$$dl_1'^2 = -g(t',0)dx_1^2$$
$$dl_2'^2 = -g(t',0)dy_2^2$$

Look now at local expansion:

$$\frac{dl_1'}{dl_1}, \quad \frac{dl_2'}{dl_2}$$

This cannot depend on orientation otherwise expansion anisotropic; nor can it depend on our position as space homogeneous. Hence

$$g(t, x) = R^2(t) g(0, 0)$$

R called cosmic scale factor.

Choose grid lines to lie along r, θ, ϕ axes. Consider two dimensional spatial surface with $\theta = \pi/2$.

Recall Gaussian curvature k:

$$k = \frac{1}{2rg^2(r)} \frac{\partial g(r)}{\partial r}$$

Problem: If this is constant independent of r, show that

$$g(r) = \frac{1}{1 - kr^2}$$

where constant, curvature scalar, selected to give unity at origin.

Robertson-Walker metric

$$ds^2 = c^2 dt^2 - R^2(t)\{\frac{1}{1-kr^2}dr^2$$
$$+r^2 \sin^2\theta\ d\phi^2 + r^2 d\theta^2\}$$

Choose $k = \pm 1$, or 0.

$k = 1$ space has positive curvature like sphere. Limit on r is 1 else proper length has no meaning $\rightarrow$ space bounded and has finite volume. Look at surface $\theta = \pi/2$ and transform r as

$$r = \sin\chi$$
$$dl^2 = d\chi^2 + \sin^2\chi d\phi^2$$

This is same as surface of globe.

Note $0 < \chi < \pi, 0 < \phi < 2\pi$ clearly satisfies the CP.

$k = -1$ leads to negative curvature, space unbounded and has infinite volume. Surface with $\theta = \pi/2$ cannot be embedded in 3 dimensions.

$k = 0$ is flat Euclidean space.

Cosmological red shift

Aim: to derive red shift due to expansion of Universe

Consider light travelling along a null radial geodesic from (ct_e, r_g) to $(ct_o, 0)$. Then from Robertson-Walker metric, where $ds = 0$ for null geodesic,

$$\int_0^{r_g} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e}^{t_o} \frac{cdt}{R(t)} =$$

$$\int_{t_e+2\pi/\omega_e}^{t_o+2\pi/\omega_o} \frac{cdt}{R(t)}$$

$$\frac{1}{R(t_o)} 2\pi/\omega_o = \frac{1}{R(t_e)} 2\pi/\omega_e$$

$$\frac{\nu_e}{\nu_o} = \frac{R(t_o)}{R(t_e)}$$

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = (R(t_o) - R(t_e))/R(t_e)$$

Contrast with more complicated expression found from using Doppler effect:

$$\frac{\nu_e}{\nu_o} = \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}$$

$$v \approx \frac{r_g(R(t_o) - R(t_e))}{t_o - t_e}$$

Note $z \to \infty$ as $R(t_e) \to 0$, explains Olbers paradox: Universe which is homogeneous, infinitely old and infinitely large, possesses infinitely bright night sky. (problem)

Physical basis of effect is exchange of photon's energy with gravitational field. Slipher observed 41 spiral nebula between 1910-26. At least 36 red-shifted $-0.001 < z < .006$. Blue shift is Andromeda!

Wirtz (1918) suggested general recession of galaxies; Hubble (1929) announced linear relation between 'velocity' and distance.

Red-shift distance relation

Aim: to relate red shift to distance of object

All the information about distant galaxies comes from measurements of their red-shifts and energy output. Two quantities related to latter are apparent and absolute luminosities.

Apparent luminosity, l, is flux of energy of star across unit proper area of telescope. Total energy output of star per second is absolute luminosity L.

Rather than use l and L, astronomers define apparent m and absolute magnitudes M:

$$l = 10^{-2m/5} 2.5 \times 10^{-8} J m^{-2} s^{-1}$$
$$L = 10^{-2M/5} 3.02 \times 10^{38} J/s$$

Problem: Show that in Newtonian theory

$$l = \frac{L}{4\pi d_L^2}, \quad d_L = 3.26 \times 10^{1+(m-M)/5} \ \text{lt yrs}$$

We now want to derive relation between apparent and absolute luminosities taking gravitation into account.

In GR: proper surface area of sphere of proper radius $rR(t)$ is

$$R^2(t)r^2 \int \sin\theta d\theta d\phi = 4\pi R^2(t)r^2$$

Fraction of photons released at $r = 0$, in time δt_e, crossing unit proper area (at observer), at proper distance $r_g R(t_o)$, and observed in time δt_o, is

$$\frac{L\delta t_e}{4\pi r_g^2 R^2(t_o)}$$

These photons are red-shifted by expansion of Universe. Hence energy reduced by factor

$$R(t_e)/R(t_o)$$

Thus power crossing unit area of observer:

$$
\begin{aligned}
l &= \frac{L\delta t_e R(t_e)}{4\pi r_g^2 R^3(t_o)\delta t_o} \\
&= \frac{L}{4\pi r_g^2}\frac{R^2(t_e)}{R^4(t_o)} \\
&\equiv \frac{L}{4\pi d_L^2} \\
d_L &= \frac{r_g R^2(t_o)}{R(t_e)}
\end{aligned}
$$

where d_L is luminosity distance.

Red-shift distance relation.

Next we derive relation between d_L and red shift z.

To do this we write

$$d_L = r_g R(t_o) \frac{R(t_o)}{R(t_e)} = (1+z) r_g R(t_o)$$

We will show that $r_g R(t_o) \sim z$ for small z and this is the limit we will investigate.

Introduce Hubble constant and deceleration parameter:

$$R(t) = R(t_o)\{1 + H_o(t-t_o) - \frac{1}{2} q_o H_o^2 (t-t_o)^2 + ...\}$$

We suppose for all observations $H_o(t_0 - t) << 1$ and hence

$$z = \frac{R(t_o)}{R(t_e)} - 1 =$$

$$H_o(t_o - t_e) + (1 + \frac{1}{2}q_o)H_o^2(t_o - t_e)^2 + ..$$

$$Inverting\ this$$

$$H_o(t_o - t_e) = z - (1 + \frac{q_o}{2})z^2 + ...$$

Problem: show this follows.

Now, need to write $r_g R(t_o)$ in terms of z.

$$\int_0^{r_g} \frac{dr}{\sqrt{1 - kr^2}} = c \int_{t_e}^{t_o} \frac{dt}{R(t)}$$

$$= \frac{c}{R(t_o)} \{t_o - t_e + \frac{H_o}{2}(t_o - t_e)^2 + ...\}$$

$$= r_g + O(r_g^3)$$

As $r_g << 1$ for current observations. Hence

$$r_g = \frac{c}{H_o R(t_o)} \{z - (1 + \frac{q_o}{2})z^2 + ... + \frac{1}{2}z^2 + ..\}$$

$$= \frac{c}{H_o R(t_o)}(z - (1 + q_o)\frac{z^2}{2} + ...)$$

$$Hence$$

$$d_L = \frac{c(1+z)}{H_o}\{z - (1+q_o)\frac{z^2}{2} + ...\}$$
$$= \frac{cz}{H_o}\{1 + (1-q_o)\frac{z}{2} + ...\}$$

Problem: show this follows.

Hence graph of d_L against z has slope c/H_o and curvature related to q_o. Examples:

$$H_o^{-1} = 13 \times 10^9 yrs, \quad q = 1.2$$

3C295 in Boötes has $z = 0.46$.

z	$t_o - t_e$	$R(t_o)/R(t_e)$	$r_g R(t_o)$	d_L
0.1	1.3$\times 10^9$	1.1	1.3$\times 10^9$	1.3$\times 10^9$
0.46	1.6$\times 10^9$	1.46	2.3$\times 10^9$	5$\times 10^9$

Problem: Evaluate $z, t_0 - t_e, R(t_0)/R(t_e), r_g R(t_o), d_L$ for a galaxy with $z = 1$. What happens for more distant objects?

Proper radial distance is $r_g R(t_0)$ and would be measured by succession of observers sending light pulses to next neighbour all at time t_o. Although impractical, proper distance is more closely related to luminosity distance than $c(t_o - t_e)$.

Observations of $d_L(z)$ require many corrections *eg.* galactic rotation and absorption.

Hubble (1918): 18 nearest galaxies: $H_o^{-1} = 2 \times 10^9$ yrs.

Sandage: $H_o^{-1} = 13 \times 10^9$ yrs, $q_o = 1.2 \pm 0.4$.

Quasars (3C273) seen optically and by radio telescopes, $z = 0.158 \rightarrow d_L = 2 \times 10^9$ yrs. However, $m = 13$, and as

$$d_L = 3.26 \times 10^{1+(m-M)/5}$$

absolute magnitude is $M = -27$ which is brighter than galaxy $M \approx -22$.

However Polamer gives angular diameter $< 0.5''$ which implies diameter 2×10^4 lt..yrs. This is tiny!

Others have been found with wide range of $d_L(z)$ values.

Friedmann equations

Correct procedure is to use Einstein field equations to derive metric but get similar answer from Newtonian theory.

Consider forces on galaxy at proper distance $rR(t)$. From Gauss' equation, these are due to matter 'interior' to galaxy. Note interior mass is $4\pi\rho r^3R^3/3$. If we assume an energy conservation equation, then:

$$\frac{1}{2}mr^2\dot{R}^2 - \frac{4\pi G}{3}\rho mr^2R^2 = -\frac{1}{2}kmr^2$$

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -k \qquad (A)$$

This equation also comes from Einstein's field equation where it turns out that k can only be ± 1, or 0.

Consider increase in energy in region of Universe due to radiation crossing surface of region. This is

$$\delta(\rho c^2 R^3)$$

and changes because of work done by pressure

$$-p\delta R^3.$$

Hence

$$\frac{d\rho c^2 R^3}{dt} = -3R^2 p \frac{dR}{dt}$$

$$\dot{\rho} = \frac{-3\dot{R}}{R}(p/c^2 + \rho) \qquad (B)$$

These are called Friedmann equations. Require equation of state $p = p(\rho)$.

Is $k > 0$? From A

$$(\frac{3H_o^2}{8\pi G} - \rho_o) = -\frac{3k}{8\pi G R_o^2}$$

Hence $k > 0$ if

$$\rho_o > \rho_c = 3H_o^2/8\pi G$$

However, differentiate A and use B

$$\ddot{R} = \frac{-4\pi G R}{3c^2}(c^2\rho + 3p)$$

$$-R_o q_o H_o^2 = \frac{-4\pi G R_o}{3c^2}(\rho_o c^2 + 3p_o)$$

$$Now\ from\ \ \rho_o = \frac{3}{8\pi G}(k/R_o^2 + H_o^2)\ \ C$$

$$p_o = \frac{-c^2}{8\pi G}\{k/R_o^2 + H_o^2(1 - 2q_o)\}$$

$$Thus\ \ k > 0\ \ iff\ H_o^2(2q_o - 1) > 8\pi G p_o/c^2$$

Problem: Draw curve of $R(t)$ versus t recalling $R(t)$ cannot be negative and $\ddot{R} < 0$. Does this vanish at some t?

Recent observations of H_o give:

$\rho_c = 1.1 \times 10^{-26} kg\ m^{-3}$

Compare proton mass 1.6×10^{-27} kg

Later on we find $p_o = 0 \rightarrow k > 0$, if $q_o > 1/2$.

Also

$$k = R_o^2 H_o^2 (2q_o - 1) \qquad D$$

Substitute into (C):

$$\rho_o = 2 q_o \rho_c$$

Hence if we use $q_o = 1.2$, then ρ_o should be $\approx 2 \times 10^{-26}$ kg m^{-3}.

Problem is measurements of q_o imply $k > 0$ but measurement (?) of ρ_o implies $k < 0$.

Finally note, that acceleration is always negative and this implies that at some time in the past $R(t) = 0$! Hence Universe had a beginning.

Evolution of $R(t)$

If matter dominated, then $p = 0$ and $\rho R^3 = constant$.

$$\dot{R}^2 - \frac{8\pi G \rho_o R_o^3}{3R} = -k$$

For large R, second term negligible and solution only possible for $k = -1$ or $k = 0$.

If $k = 1$, no solution as $r \to \infty$ showing $R(t)$ bounded above.

$$R_{max} = \frac{8\pi G}{3}\rho_o R_o^3$$

Weighing the Universe

Close-by galaxy of radius a rotates with angular velocity v/a thus for star on rim:

$$\frac{MG}{a^2} = \frac{v^2}{a}, \quad M = \frac{v^2 a}{G}$$

Measure distribution of red-shifts from galaxy; use Döppler effect to give estimate of v. Combine with angular size measurements yielding a to give M. Use red-shift to give distance of galaxy and apparent luminosity to yield absolute luminosity L.

Find M/L typically to be:

$$\begin{aligned} M/L &= \zeta M_{sun}/L_{sun} \\ 1 < \zeta < 20 &\quad \textit{Spiral galaxies} \\ \zeta \approx 50 &\quad \textit{elliptical galaxies} \\ \zeta \approx 21 &\quad \textit{average} \end{aligned}$$

Observe number density of spirals and elliptical galaxies and combine with ζ values to give galactic density finding $\rho_g = 3.1 \times 10^{-28}$ kg m^{-3} = 0.028 ρ_c. If $\rho_o = \rho_g \rightarrow q_o = 0.014$ or Universe has $k = -1$ *ie.* open.

Either measured q_o or ρ_o is wrong!!

Radiation mass in microwave background negligible as $\rho_r c^2 = E/V = aT^4$ where a proportional to Stefan-Boltzmann constant. Hence $\rho_r = 10^{-31}$ kg m^{-3}.

Evolution of R in matter dominated epoch

From FE and C and D:

$$\rho R^3 = \rho_o R_o^3$$

$$\dot{R}^2 + k = \frac{8\pi G}{3}\frac{\rho_o R_o^3}{R}$$

$$\rho_o = 2q_o\rho_c = 6q_o H_o^2/8\pi G$$

$$\left(\frac{\dot{R}}{R_o}\right)^2 = H_o^2(1 - 2q_o + 2q_o\frac{R_o}{R})$$

$$H_o t = \int_0^{R/R_o} \frac{1}{\sqrt{1 - 2q_o + 2q_o/x}} dx$$

$$t_o = \frac{1}{H_o}\int_0^1 \frac{1}{\sqrt{1 - 2q_o + 2q_o/x}} dx < 1/H_o$$

If $q_o > 1/2$ or $k = 1$, put

$$\frac{R}{R_o} = x = \frac{2q_o}{2q_o - 1} \sin^2(\theta/2),$$

$$\text{then } H_o t = \frac{q_o}{(2q_o - 1)^{3/2}} (\theta - \sin\theta)$$

which is equation of cycloid.

Problem: Sketch $R(t)/R_o$ as function of t.

$R(t)$ is maximum when $t = t_m = \frac{1}{H_o} \frac{\pi q_o}{(2q_o - 1)^{3/2}}$ For $q_o = 1.2$, $t_m = 2.27/H_o \sim 30 \times 10^9$ yrs.

$$R(t_m) = \frac{2q_o R_o}{2q_o - 1}$$

Then $R(t) \to 0$ as $t \to 2t_m$.

Present age of Universe found when $x = 1$ or $\theta = 99^\circ$ and $H_o t_o = 0.54$ or $t_o = 7 \times 10^9$ yrs.

If $k = 0$, then $\rho = \rho_c, q = 1/2$ and $R(t) = R_o(\frac{3H_o t}{2})^{2/3}$. Present age $t_o = \frac{2}{3} H_o^{-1} \approx 9 \times 10^9$ yrs.

If $k = -1$, then $q_o < 1/2, \rho_o < \rho_c$

$$x = \frac{2q_o}{1 - 2q_o} \sinh^2(\psi/2)$$
$$H_o t = \frac{q_o}{(1 - 2q_o)^{3/2}} (\sinh(\psi) - \psi)$$

Problem: Sketch behaviour of $R(t)$ in this case.

$R(t) \rightarrow \infty$ with t. If $\rho_o = \rho_g$, then $t_o = 0.96 H_o^{-1} = 12.5 \times 10^9$ yrs.

Age of Universe

Age of Earth $\approx 4.5 \times 10^9$ yrs.

Age of Galaxy (Burbidge, Fowler, Hoyle, 1957)

Helium formed in big bang; heavier elements formed subsequently in stars by rapid neutron addition; At time of formation of star, concentrations and decay rates:

$$\frac{[U^{235}]}{[U^{238}]} \approx 1.65 \pm 0.15$$

$$\lambda(U^{235}) = 0.971 \times 10^{-9} yr.^{-1}$$

$$\lambda(U^{238}) = 0.154 \times 10^{-9} yr.^{-1}$$

$$Present\ conc.$$

$$\frac{[U^{235}]}{[U^{238}]} \approx 0.00723$$

Variation is

$$\frac{[U^{235}]}{[U^{238}]} = \frac{[U^{235}]_0}{[U^{238}]}_0 \exp(-(\lambda(U^{235} - \lambda(U^{238})t)$$

$$0.00723 = 1.65 \exp(-0.817 \times 10^{-9} t)$$

Problem: Show this follows from radioactive decay rates.

Hence, present age of galaxy is

$$\approx 6.6 \times 10^9 yrs.$$

$$Hence\ q_o < 2.3$$

If developed earlier, would have greater concentrations of fissile U^{235} !

Horizons

Light signals

$$\int_0^{r_g} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e}^{t_o} \frac{cdt}{R(t)}$$

Limit on t_e is 0. Hence limit on r_g as light has not yet reached us. Called particle horizon at proper distance

$$d_H = R(t_o) \int_0^{r_g} \frac{dr}{\sqrt{1-kr^2}} =$$

$$R(t_o) \int_{t_e}^{t_o} \frac{cdt}{R(t)}$$

$$For\ k = 1$$

$$= \frac{2c}{H_o\sqrt{(2q_o-1)}} \cos^{-1} \sqrt{\{1 - \frac{2q_o-1}{q_0}\}}$$

This is 19×10^9lt. yrs. for $q_o = 1.2$.

Proof:

$$R(t) = \frac{R_o 2q_o \sin^2(\theta/2)}{2q_o - 1}$$

$$H_o t = \frac{q_o}{(2q_o - 1)^{3/2}}(\theta - \sin\theta)$$

$$H_o dt = \frac{q_o}{(2q_o - 1)^{3/2}}(1 - \cos\theta)d\theta$$

$$d_H = \frac{c}{H_o\sqrt{2q_o - 1}} \int d\theta (\frac{1 - \cos\theta}{2\sin^2(\theta/2)})$$

$$= \frac{c\theta}{H_o\sqrt{2q_o - 1}}$$

$$Now \ \ \cos^2(\theta/2) = (1 - \frac{2q_o - 1}{2q_o})\frac{R}{R_o}$$

$$d_H = \frac{2c}{H_o\sqrt{2q_o - 1}} \cos^{-1} \sqrt{(1 - \frac{2q_o - 1}{2q_o})}$$

Cosmic microwave background

Photons absorbed by electrons promoting transitions and leading to equilibrium involving creation and annihilation of photons.

Boltzmann distribution:

$$p_r = \frac{1}{Z} \exp(-E_r/kT)$$
$$Z = \sum_r \exp(-E_r/kT)$$

In nature particles are either bosons or fermions. Bosons possess values of S which are integers. Electrons, neutrons and protons having spin $S = 1/2$ are fermions while photons are bosons with spin 1.

The wavefunction of a boson has to be symmetric upon interchange of particles while that for Fermions is antisymmetric, ie for two particles:

$$\Psi(r_1, s_1, r_2, s_2) = -\Psi(r_2, s_2, r_1, s_1)$$

Crucial to note only one fermion can be put into a state but number of bosons in state is unlimited. If n_r photons occupy a state r with energy $n_r E_r$, then probability of finding this state is then $exp(-n_r E_r/kT)/Z$ where

$Z = \sum_{n_r} exp(-n_r E_r/kT)$.

Since n_r is 0, 1, 2, .. mean number of photons in state is

$$\sum_{n_r} n_r exp(-n_r E_r/kT)/Z$$

$$= \frac{1}{e^{E_r/kT} - 1}$$

Energy levels of photon in a cubical cavity with side L are

$$E_r = \hbar\omega_r = \hbar c|\mathbf{k}_r|$$

$$\mathbf{k}_r = \frac{2\pi}{L}(n_x, n_y, n_z)$$

There are two possible states corresponding to polarisation of photon.

Thus mean number of photons in state r is

$$\langle n_r \rangle \;= \frac{1}{e^{\hbar\omega_r/kT} - 1}$$

Mean energy density of black body radiation

$$E = \frac{1}{V}\sum_r \langle n_r \rangle \; \hbar\omega_r$$

$$= \frac{2}{V}\sum_{\mathbf{k}_r} \frac{\hbar\omega_r}{e^{\hbar\omega_r/kT} - 1}$$

$$\frac{2}{8\pi^3}\int d\mathbf{k} \; \frac{\hbar ck}{e^{\hbar ck/k_B T} - 1}$$

$$= aT^4$$

$$a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$$

$\sigma = ac/4 = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is called Stefan-Boltzmann constant

Can be shown that energy carried through unit area of surface of black body in 1 sec is $Ec/4V = acT^4/4 = \sigma T^4$.

This shows that a star of twice the temperature as another radiates 16 times as much energy.

For non-equilibrium, when temperature varies with T, then

$$\langle n_r(t)\rangle \ = \frac{1}{e^{\hbar\omega_r/kT(t)} - 1}$$

$T(t)$ is matter temperature.

Early Universe

Aim: to derive temperature of microwave background

Early Universe composed of hot plasma; strong coupling between photons and matter. As R increases, photons red-shifted and T decreases.

When $kT \approx 1$ eV, $t = t_r, T = T(t_r) \approx 10^4$ K, electrons recombine with protons to give neutral H. Universe becomes transparent to photons whose number, $\propto n_p R^3$, remains constant. Energy density of photons:

$$
\begin{aligned}
aT^4 &= \rho_r c^2 \\
&= n_p \hbar \omega_k \propto \frac{1}{R^3}\frac{\hbar c}{R}
\end{aligned}
$$

This shows that after the time $t = t_r$, $T \propto 1/R$ and as

$$n_k(t_o) = \frac{1}{e^{\hbar\omega_k/kT(t_o)} - 1}$$

we find that

$$T_o = T(t_r)\frac{R(t_r)}{R(t_o)}$$

Variation of $T(t)$ for early Universe

This is fireball era when matter was dominated by radiation. This means equation of state is different. For a photon gas:

$$\rho c^2 = aT^4$$

$$p = E/3V = aT^4/3$$

$$2^{nd}\ Friedmann\ eqn\ gives$$

$$\frac{d(T^4R^3)}{dR} = -R^2T^4$$

$$\rightarrow TR = constant$$

Hence for times $t << t_r$, TR is also constant.

This is same relation for matter dominated Universe but for different reasons. Of course the constants are not necessarily the same but we assume they are and hence take

$$T(t) = \frac{R(t_o)T(t_o)}{R(t)}$$

for all times

Gamow' s argument for $T(t_o) = T_o$

In early Universe, proton and neutrons combined to form deuterons plus γ rays (high energy photons) as well as deuterons which subsequently formed helium.

Binding energy of deuteron $\approx 10^5$eV or $10^9 K$. If $T = 10^9 K$, need a proper density of protons and neutrons to be $n = 10^{18}$ cm^{-3}, so that 50% fuse into deuterium. The excess energy is liberated as photons.

Now ρ_c now corresponds with roughly 1 proton per m^3. For matter dominated Universe:

$$\frac{R(t)}{R_o} = \Big(\frac{n_o}{n(t)}\Big)^{1/3} \approx \Big(\frac{10^{-6}}{10^{18}}\Big)^{1/3} = 10^{-8}$$

Hence $T_o \approx 10$K.

There should be a gas of photons with this temperature left over from the big bang.

Discovery of comic microwave background

Penzias and Wilson (1966) accidentally observed this. Noticed weak background signal at $\lambda = 7.35$ cm in antenna used for Echo satellite.

$$T_A(\theta) = 4.4 + 2.3 \ \sec(\theta) K$$

Second term is related to absorption through atmosphere $\propto$ $sec(\theta)$. After various corrections $T_o = 3.5K$.

Really need proof that black body distribution is present. The energy density per unit frequency is maximum for photons with wavelengths of about 0.1 cm.

Cyanogen molecules in outer space between us and ζ Ophiuchi absorb long wave length radiation. Ground state has molecular rotations $J = 0, 1$ separated by 0.264 cm. Electronic excited states also possess a rotational doublet so we get two transitions at 3874.6 Å and 3875.16 Å corresponding to $\Delta J = 1$. Both lines observed in absorption because of microwave background excited $J = 1$ electronic ground state; gives $T = 2.3K$.

Experiment also gives anisotropy is less than few %.

Early Universe

At the moment we live in matter dominated universe with

$$\rho_r c^2 = 10^{-31} kg/m^3 \propto 1/R^4$$
$$\rho_c c^2 = 10^{-26} kg/m^3 \propto 1/R^3$$

But this becomes radiation dominated when $R(t)/R(t_o) < 10^{-5}$ or $T(t) > 10^5 T(t_o) \approx 10^5 K$.

To find out when this happened, need to know variation of $R(t)$ versus t. For $k = 0$ in matter dominated case

$$R(t) = R(t_o)(\frac{3H_o t}{2})^{2/3}$$

Hence radiation dominated $t < 10^3 yrs$

However, in early Universe, $R(t)$ has different t dependence because no longer matter dominated.

To find time dependence need to solve Friedmann first eqn with $\rho = \rho_r$.

Gravitation term dominates k in this equation. Hence choose $k = 0$.

$$\dot{R}^2 = \frac{8\pi G \rho R^2}{3}$$
$$\rho = \rho_r = aT^4/c^2$$
$$R = R_0 T_0 / T$$

Hence

$$\frac{\dot{R}}{R} = \sqrt{\frac{8\pi G a T^4}{3c^2}}$$
$$-\frac{\dot{T}}{T^3} = \sqrt{\frac{8\pi G a}{3c^2}}$$
$$\frac{1}{T^2} = \sqrt{\frac{32\pi G a}{3c^2}} t$$

Finally,

$$t = \sqrt{\frac{3c^2}{32\pi G a}} T^{-2} \approx \left(T/10^{10}K\right)^{-2}$$

Early Universe- He production

$T > 10^{12}K, \quad t < 10^{-4}s, \quad kT > 0.1GeV$. Recall $m_pc^2 = 0.9GeV$, hence anti-protons and anti-neutrons are not present. Instead

$\gamma, e^-, e^+, \nu, \overline{\nu}, \mu, \overline{\mu}$.

Recall $m_\mu c^2 = 0.1GeV, m_e c^2 = 0.5MeV$.

$T \approx 10^{12}K$, $\mu, \overline{\mu}$ were annihilated as

$$\mu^+ \rightarrow e^+ + \nu_e + \overline{\nu_\mu}$$

$$\mu^- + p \rightarrow \nu_\mu + n$$

$T \approx 5 \times 10^9 K, \quad kT \approx 0.5MeV, \quad t \approx 4s, \quad e, e^-$ began to annihilate producing γ which eventually became part of micro-wave background. Also, n decayed via weak interactions and leaving $n : p$ of about $1 : 5$.

$T < 10^9 K, \quad t > 3$ min:

$$n + p \rightarrow He$$

with 27% of nucleons turned into He.

$p + e \rightarrow H, \quad T \approx 4000K$, photons decoupled from matter.

Define fraction of nucleons in nuclide i to be $x_i = n_i A_i / n_N$.

If neutrons and protons in equilibrium with each other via weak interactions:

$$n \rightarrow p + e^- + \nu$$

then:

$$\begin{aligned} x_n = (x_p + x_n)e^{-(m_n - m_p)c^2/kT} \\ = (x_p + x_n)e^{-b/kT} \end{aligned}$$

Here $b = 1.293$ MeV or $1.5 \times 10^{10} K$. At $T = 10^{10} K, \quad x_n \approx 0.18$. At $T = 10^9 K, \quad x_n \approx 10^{-6}$.

and very few free neutrons left to form He or indeed any other element!!!.

However this weak decay is very slow and rate-constant at this temperature is about 1013 secs. Under these conditions, other reactions take place. Instead those neutrons present at $10^{10}K$ are converted into deuterons and hence into He.

$$n + p \rightarrow d + \gamma,$$
$$d + d \rightarrow^3 H + p$$
$$^3H + d \rightarrow^4 He + n$$

So all free neutrons at $T \approx 10^9 K$ converted into He.

To determine precise concentrations, need to carefully solve rate equations.

At $T = 10^{10}K, \quad x_n \approx 0.18$ and as all these are converted into He, we conclude 36% of nucleons are in form of He. In fact a more detailed calculation gives 27% close to observed helium abundance.

Element abundance: H, He, C, N, O and Ne are common elements and all others are rare. Gamow noted that big bang could lead to He production by neutron addition (nucleosynthesis) but not other elements as no stable nuclide with $A = 5$. Wagoner, Fowler and Hoyle (1967) pointed out that other elements could be formed in stars where T much higher than in big-bang when nucleosynthesis was occurring.

Large Red shifts

Take $k = 0, q = 1/2$ and hence $R(t) = R(t_0)(3H_0t/2)^{2/3} = R(t_0)(t/t_0)^{2/3}$.

$$t_0 = 2/3H_0$$

$$d_L = r_g(1+z)R(t_0)$$

$$r_g = c\int \frac{dt}{R(t)} = (\frac{3t_0c}{R(t_0)})(1-(t_e/t_0)^{1/3})$$

$$1+z = (t_0/t_e)^{2/3}$$

$$t_e/t_0 = (\frac{1}{1+z})^{3/2}$$

$$d_L = r_g(1+z)R(t_0) = (3t_0c(1+z))(1-(t_e/t_0)^{1/3})$$

$$= (\frac{2c(1+z)}{H_0})(1-(\frac{1}{1+z})^{1/2})$$

For $z = 6$, $d_L = 1.24c/H_0$ or objects are 16×10^9 lt yrs if $H_0^{-1} = 13 \times 10^9$ lt yrs.

Time taken to reach here is:

$(1 - 0.053)t_0 =$ 0.95 $t_0 = 8.2 \times 10^9$ lt yrs.

Problem: what is proper distance to objects?

Lemaitre Cosmology

Recall FE:

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -k$$
$$\frac{d}{dt}(\rho c^2 R^3) = -3R^2 p\frac{dR}{dt}$$

These lead to decelerating universe as $\ddot{R} = \frac{-4\pi GR}{3c^2}(c^2\rho + 3p)$ and is less than zero.

Last implies that static universe impossible. Only by changing Einstein field equations can we find a static solution.

This can be done by introducing a force which attracts particles of matter and leads to an energy - called dark energy. This results in equations written in terms of modified density and pressure

$$\tilde{p} = p - \frac{\lambda}{8\pi G}$$

$$\tilde{\rho} = \rho + \frac{\lambda}{8\pi c^2 G}$$

Here λ is called the cosmological constant. Hence FE become

$$\dot{R}^2 - \frac{8\pi G}{3}\tilde{\rho}R^2 = -k \qquad (A)$$

$$\frac{d}{dt}(\tilde{\rho}c^2R^3) = -3R^2\tilde{p}\frac{dR}{dt} \quad (B)$$

$$\frac{d}{dt}(\tilde{\rho}c^2R^3) = \frac{d}{dt}(\rho c^2R^3) + 3\Big(\frac{\lambda R^2\dot{R}}{8\pi G}\Big)$$

Hence

$$\frac{d}{dt}(\rho c^2R^3) = -3R^2p\dot{R}$$

Note for matter domination $p = 0$ and hence ρR^3 is constant as before (matter conservation).

The acceleration is

$$\ddot{R} = \frac{-4\pi GR}{3c^2}(c^2\tilde{\rho} + 3\tilde{p})$$

$$= \frac{-4\pi G R}{3c^2}(c^2\rho + 3p - 2\frac{\lambda}{8\pi G})$$

RHS can now be positive if $\lambda > 0$ and big enough. For matter domination, ρ is large initially but decreases as $1/R^3$. Hence cosmological constant unimportant for early universe. Eventually, as ρ decreases, λ term dominates and

$$\ddot{R} \sim \frac{\lambda}{3c^2}R$$

whose solution is $R = R_0 \exp(H(t - t_0))$ where $H^2 = \frac{\lambda}{3c^2}$

There is a static solution (which is unstable) found by setting $\dot{R} = \ddot{R} = 0$.

$$c^2\rho + 3p - 2\frac{\lambda}{8\pi G} = 0$$

and from Friedmann equation A

$$\frac{8\pi G}{3}\tilde{\rho}R^2 = k$$

$$\text{or} \quad \frac{8\pi G R^2}{3}(\rho + \frac{\lambda}{8c^2\pi G}) = k$$

For matter domination, $p = 0$ and $\rho c^2 = 2\lambda/8\pi G$ and $R^2\lambda/c^2 = k$ Hence $k > 0$ and must equal 1 from which the size of Universe is $R = c/\sqrt{\lambda}$.

To investigate evolution, of accelerating universe, set $\rho R^3 = \alpha c/(4\pi G\sqrt{\lambda})$ so that $\alpha = 1$ corresponds to static solution. Set $R = xc/\sqrt{\lambda}$ to scale Universe wrt static case.

Then use Friedmann equation

$$\begin{aligned}
\dot{R}^2 = \frac{8\pi G}{3R}\Big(\rho R^3 + \frac{\lambda R^3}{8\pi G c^2} - \frac{3kR}{8\pi G c^2}\Big) \\
= \frac{1}{R}\Big(\frac{\lambda R^3}{3} - kR + \frac{2\alpha}{3\sqrt{\lambda}}\Big) \\
\dot{x}^2 = \frac{\lambda}{3c^2 x}(2\alpha + x^3 - 3kx) \\
= \frac{\lambda}{c^2}H(x)^2
\end{aligned}$$

$$H^2 = \frac{1}{3x}(2\alpha + x^3 - 3kx)$$

$$\text{Hence} \quad t = \int_0^x \frac{dx}{H(x)}$$

Problem: Show $R(t) \sim t^{2/3}$ initially.

Problem: Show minimum value of $\dot{x}$ occurs when $x = \alpha^{1/3}$.

$R \sim t^{2/3}$ initially, then slows down so that acceleration least at $x = \alpha^{1/3}$ then speeds up again.

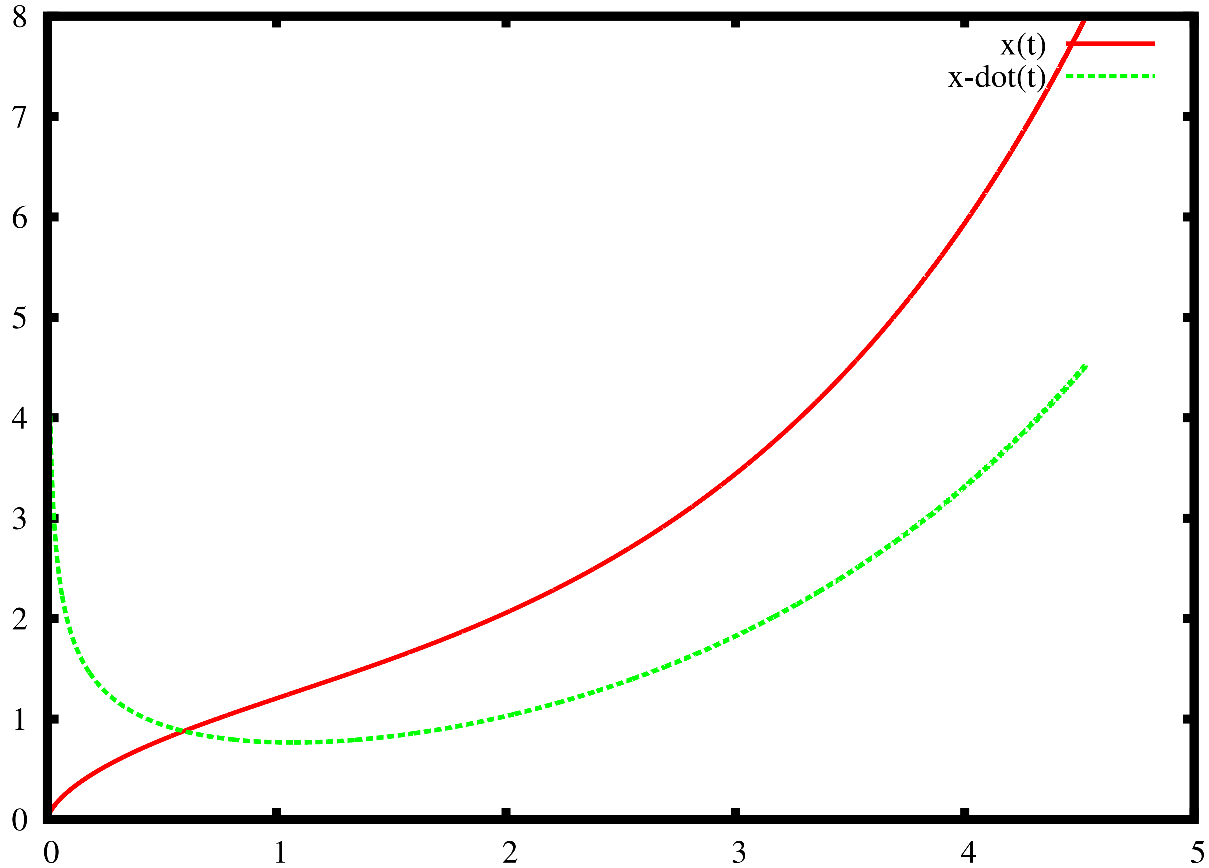
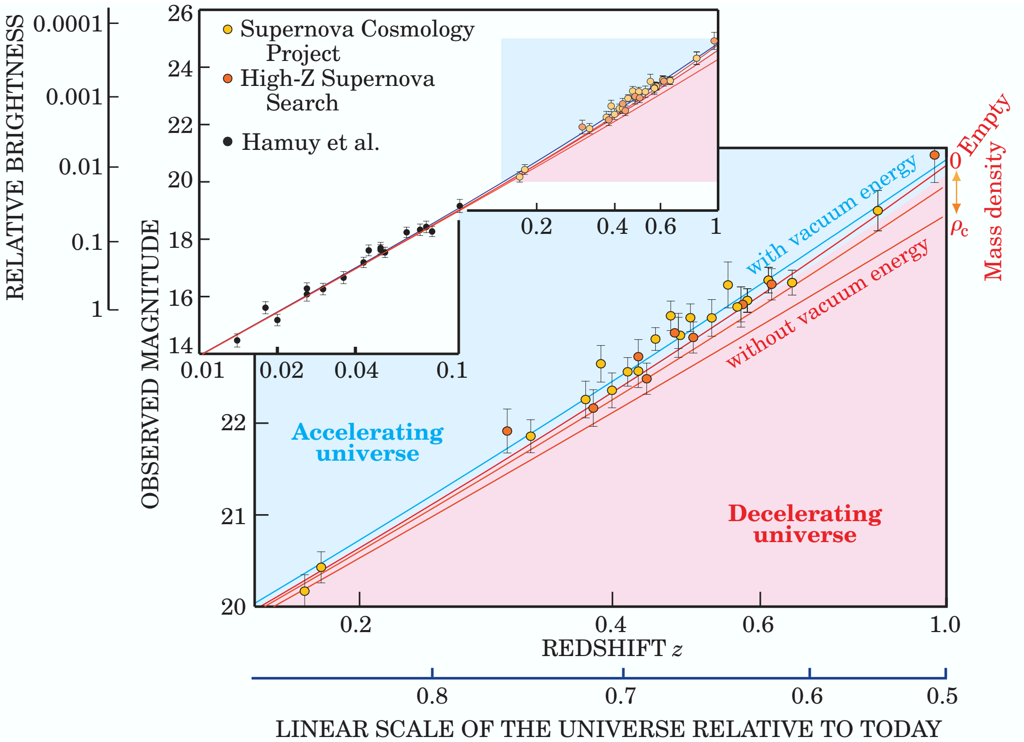


Figure 1: Lemaitre evolution, $x(t)$ and $\dot{x}(t)$, for $\lambda = 1, \alpha = 2, k = 1$, Universe expands first with deceleration, then coasts around static value, then expands exponentially.

Cosmological Constant

Figure 3. Observed magnitude versus redshift is plotted for well-measured distant[12,13] and (in the inset) nearby[7] type Ia supernovae. For clarity, measurements at the same redshift are combined. At redshifts beyond $z = 0.1$ (distances greater than about 10^9 light-years), the cosmological predictions (indicated by the curves) begin to diverge, depending on the assumed cosmic densities of mass and vacuum energy. The red curves represent models with zero vacuum energy and mass densities ranging from the critical density ρ_c down to zero (an empty cosmos). The best fit (blue line) assumes a mass density of about $\rho_c/3$ plus a vacuum energy density twice that large—implying an accelerating cosmic expansion.



From Friedmann's equation:

$$\dot{R}^2 - \frac{8\pi G R^2}{3}(\rho + \frac{\lambda}{8c^2\pi G}) + k = 0$$
$$H_o^2 - \frac{8\pi G}{3}(\rho_o + \frac{\lambda}{8c^2\pi G}) + k/R_o^2 = 0$$
$$k/R^2 = \frac{\rho_o + \rho_\lambda}{\rho_c} - 1$$

where $\rho_c = 3H_o^2/8\pi G$ as before. Observations of type I supernova suggest that $k = 0$ and $\rho_\lambda = 0.7\rho_c$ and $\rho_o = 0.3\rho_c$. The estimate of ρ_o is closer to ρ_g but this begs further questions.

Why is energy in cosmological force so great at the moment? why should ρ_o and ρ_λ to be so close at the moment! and why should $k \sim 0$.

We do not have answers to these questions!