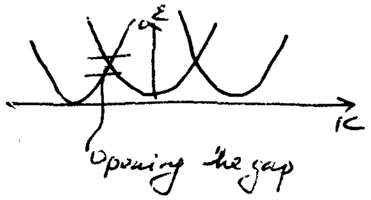
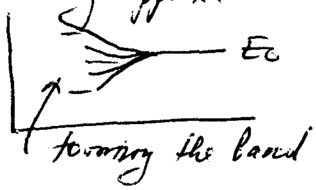
UNIVERSITY OF EXETER - SCHOOL OF PHYSICS
SOLUTION TO EXAMINATION QUESTION

Module Number	PHYM401
Year of Examination	2010
Question Number	1–2
Name of Setter	AKS
Initials of Checker	HINTS AND SOLUTIONS

(i) Free-electron model:



Tight-binding approx.



Bloch condition: in a periodic potential

$$\psi_k(x+x_j) = e^{ikx_j}\psi_k(x)$$

$$\psi_k(x+x_j) = c\sum_m e^{ikx_m}\varphi(x+x_j-x_m) = e^{ikx_j}\, c\sum_m e^{ik(x_m-x_j)} \times \varphi(x-(x_m-x_j)) = e^{ikx_j}\psi_k(x).$$

With increasing a, band width 2γ decreases while m^* increases.

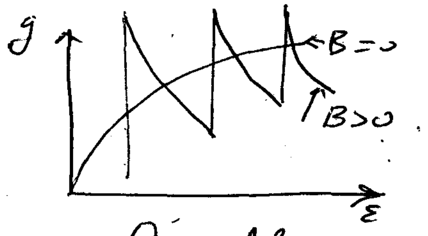
(ii) Anderson transition: tight binding with disorder. Scaling theory is checked by $R(T)$.

(i) Hall effect: $E_y = \underbrace{R_H}\, j_x B_z$ $\quad R_H = -\frac{1}{en}$, n - carrier density

For $\omega_c\tau_p < 1$ $\quad R_H$ depends not only on n, but also on μ.

In Al:
1st BZ. 2N electrons
2nd BZ $n_2 + p_2 = 2N$
3rd BZ: $n_3 = N - n_2 \longrightarrow N_{eff} = n_3 - p_2 = -N$.

(ii)



Semiclassical model: $E_F/\hbar\omega_c \gg 1$ (many Landau levels)
Then $\lambda_F = \frac{2\pi}{k_F} \ll R_c$

$$R_c = \sqrt{\frac{2\hbar}{eB}\left(n+\frac{1}{2}\right)} \qquad n = 0, 1, 2\ldots$$

$$\Phi_n = AB = \pi R_c^2 B = \frac{\pi\, 2\hbar B}{eB}\left(n+\frac{1}{2}\right) = \frac{h}{e}\left(n+\frac{1}{2}\right)$$

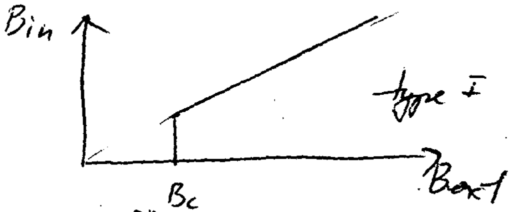
SCHOOL OF PHYSICS
SOLUTION TO EXAMINATION QUESTION

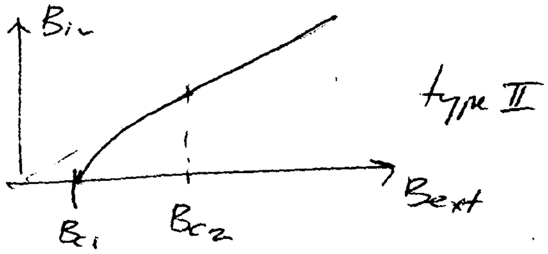
Module Number	PHYM401
Year of Examination	2010
Question Number	3-4
Name of Setter	AKS
Initials of Checker	HINTS AND SOLUTION

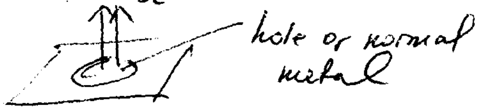
Isotope effect: $M^{\alpha}T_c = \text{const}$. $\alpha \sim 0.5$

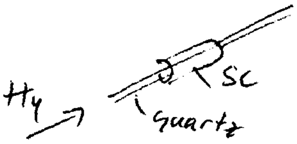
Cooper pair: Two electrons with opposite k and opposite spin, interacting by phonon emission/absorption.

Bose-Einstein statistics; at low T Bose condensation and all particles in one state.

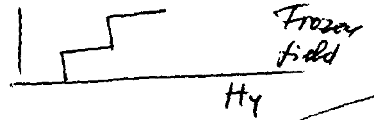




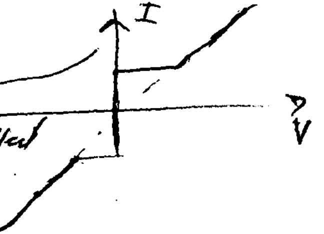




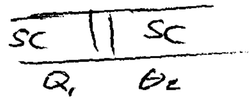
Observation in 1961.



Josephson junction:



$J = J_0 \sin(\theta_1 - \theta_2)$



---

i) $D(\omega,k) = \varepsilon_0\varepsilon(\omega,k)E(\omega,k)$ $\varepsilon(\omega,0) = 1 - \frac{\omega_p^2}{\omega^2}$, $\varepsilon(0,k) = 1 + \frac{k_s^2}{k^2}$

$\omega_p^2 = \frac{e^2 n}{\varepsilon_0 m}$; $k_s^2 = \frac{e^2}{\varepsilon_0} g(\varepsilon_F)$

waves in plasma Electrostatic screening

) $M = \chi H$, Dia: χ is T-independent
Para: $\chi \propto T^{-1}$

Small paramagnetism in metals and low e-e scattering rate are due to Pauli principle

$\chi = \frac{C}{T - T_c}$ - Curie-Weiss law, Failure: value (low) of T_c.

Exchange interaction: dependence of energy on the symmetry of W.F.

$E_{ex} = -2J_{ex}S_1S_2$
exchange integral

Ferromagnetism is due to "electrostatic" and not magnetic interaction.

PHYM421/PHYM501

UNIVERSITY OF EXETER

SCHOOL OF PHYSICS

MAY 2010

STATISTICAL MECHANICS

Hints for the Solutions

1. (a) Pay attention to the limitations imposed by the Pauli principle to the fermionic filling of the single particle states.

   (b) Use the definition of the partition function, with the energies of the many-body states deduced in the previous point.

   (c) Use the proper expansions: $e^{-\epsilon} \simeq 1 - \epsilon$ and $e^{-1/\epsilon} \ll 1$ for $\epsilon \ll 1$.

   (d) Express the free energy F from the partition function. Use the expansions of the previous point *c*), as well as $\log(1+\epsilon) \simeq \epsilon$ for $\epsilon \ll 1$.

   (e) Use the definition of the entropy from the free energy and the expansions of the previous points *c*) and *d*). Recover the correct limit of the entropy at zero temperature.

2. (i) From $dE = TdS + \mu dN - PdV$ introduce a total differential to eliminate the PdV term in favour of a term AdP. Determine A and deduce T, μ and V by means of partial derivatives.

   (ii) Use the definition of the average energy. Pay attention that this is for a single particle. Deduce the result for a massive particle in 3D, i.e. use the proper value of α.

   (iii) (a) Pay attention to the massless dispersion and the reduced dimensionality (2D).

   (b) Use the definition of the Fermi energy and pay attention to deduce k_F with the massless spectrum.

(c) Justify the sign of the chemical potential μ for non-interacting bosons (i.e. pay attention to the form of the Bose distribution). Consider the limiting value of μ where BEC occurs. For this value, impose that all the particles occupy the lowest single-particle state and deduce the corresponding critical temperature.

3. (a) Use the dimension $[k] = 1/\text{length}$.

   (b) Pay attention to the dimensionality (1D) and the power-law of the dispersions.

   (c) Use the Bose distribution and the DOS deduced in *b*) to calculate the average energy at temperature T, paying attention to the proper value of μ for phonons. In this case the Debye energy can be considered as infinite.

   (d) Use the definition of the heat capacity from the energy E. From the power-law dependences of C_V on T deduce the dominant branch at low T.

   (e) Identify the temperature at which the two C_V coincide.

4. (a) Discuss the difference between μ and E_F, and consider their behaviour at $T = 0$ and $T > 0$.

   (b) Pay attention to the reduced dimensionality (2D) and to the massive spectrum of fermions.

   (c) Give the general expression connecting the total particle number with the Fermi distribution at temperature T.

   (d) Consider the limit of the expression in *c*) for $T = 0$ and deduce k_F with the proper massive dispersion.

   (e) Discuss the sign and magnitude of μ in the ideal gas approximation. Use this to expand the Fermi distribution in *c*) and perform the integral over energy. From the result, deduce the dependence of μ on T.

PHYSICS EXAMINATION PROBLEMS
SOLUTIONS AND HINTS FOR STUDENT SELF-STUDY

Module Code	**PHYM422**
Name of module	**QUANTUM PHYSICS III**
Date of examination	**June 2010**

1. $m = \pm 1 \quad i.e. \quad E_1^{(0)} = \hbar^2/2MR^2$ and $\psi_{\pm 1} = \frac{1}{\sqrt{2\pi}}\exp(\pm i\phi)$

   1st order correction to ground-state $E_1^{(0)} = V_{nn} = \int \left|\psi_n^{(0)}\right|^2 (-eER\cos\phi)\,\mathrm{d}\phi$

   $V_{nn} = \int \frac{1}{2}(-eER\cos\phi)\,\mathrm{d}\phi = 0$ (symmetry)

   2nd order correction to ground-state $E_0^{(2)} = -Me^2E^2R^4/\hbar^2$

2. $E_\mathrm{f} = \hbar\omega - \hbar^2\gamma^2/2m$

   Matrix element $V_{\mathrm{f}i} = -\dfrac{4ik\gamma^{3/2}eEH}{\left(k^2+\gamma^2\right)^2}$

   Expression for $g(E_\mathrm{F}) = \dfrac{1}{\pi\hbar}\sqrt{\dfrac{m}{2E_\mathrm{F}}}$

   Ionization rate $\Gamma = \dfrac{e^2E_0^2\gamma^3}{m^2\omega^4}\sqrt{\dfrac{1}{2m}\left(\hbar\omega - \dfrac{\hbar^2\gamma^2}{2m}\right)} \quad \Rightarrow \quad \Gamma \sim \omega^{-7/2}$ for large ω

3. (i)

   $E_+ = 2V_0 \qquad \psi_+ = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$

   $E_- = 0 \qquad \psi_- = \frac{1}{\sqrt{2}}\left(\psi_1(x) - \psi_2(x)\right)$

   (ii) $S = 0$ requires an antisymmetric spin part of wavefunction hence the spatial part must be symmetric. Lowest energy is $E_\mathrm{a} = 2E_1$ for $|\psi_\mathrm{a}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle\right)\psi_1(x_1)\psi_1(x_2)$

   $S = 1$ requires a symmetric spin part so spatial part is antisymmetric and particles must occupy different energy levels. All three such states have energy $E_1 + E_2$ so $S = 0$ is the groundstate.

   $S = 0$ is not split by magnetic field, but $S = 1$ has its three-fold degeneracy lifted so when $2\mu B > E_2 - E_1$ the energy of the state with $S_z = 1$ has fallen enough for it to become the new groundstate.

4. $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \dfrac{m^2U_0^2a^6}{(2\pi)^5\hbar^4}\exp\left(-4k^2a^2\sin^2(\theta/2)\right)$

   $\sigma_\mathrm{tot} = \dfrac{4\pi m^2U_0^2a^6}{(2\pi)^5\hbar^4}$

PHYSICS EXAMINATION PROBLEMS
SOLUTIONS AND HINTS FOR STUDENT SELF-STUDY

Module Code	**PHYM423**
Name of module	**Classical and Quantum Fluids**
Date of examination	**January 2010**

1. The normal boiling point of ^{4}He is 4.2 K so

$$p_0 = 1\text{ bar}\times\exp\left(\frac{7.2\text{ K}}{4.2\text{ K}}\right) = 5.5\text{ bar and } p(100\text{ mK}) = p_0\times\exp\left(\frac{-7.2\text{ K}}{100\text{ mK}}\right) = 3\times10^{-31}\text{ bar}$$

$$pV = nk_\text{B}T \quad\text{so}\quad \frac{n}{V} = \frac{3\times10^{-31}\text{ bar}\times10^{5}\text{ N m}^{-2}\text{ bar}^{-1}}{1.38\times10^{-23}\text{ J K}^{-1}\times0.1\text{ K}} = 0.02\text{ m}^{-3}$$

2. (ii) Turbulent layer thickness $\delta \propto x^{0.8}$, where x is the length of the train.

$$v_\text{m}= 180\text{ km h}^{-1} = 50\text{ m s}^{-1} \rightarrow \delta = 0.37\left(\frac{\nu}{\text{v}_\text{m}}\right)^{0.2} x^{0.8} = 0.37\left(\frac{1.5\times10^{-5}}{50}\right)^{0.2} 100^{0.8} = 0.73\text{m}$$

For flat plate $C_D = C_\text{F} = \dfrac{2F_\text{D}}{\rho\,\text{v}_\text{m}^2 xy}$, where y is the breadth of the plate and F_D the drag force.

$$0.074Re^{-0.2} = \frac{2F_\text{D}}{\rho\,\text{v}_\text{m}^2 xy} \Rightarrow F_\text{D} = \frac{0.074Re^{-0.2}\rho\,\text{v}_\text{m}^2 xy}{2} = \frac{0.037\text{v}_\text{m}^{-0.2}x^{-0.2}\rho\,\text{v}_\text{m}^2 xy}{\nu^{-0.2}}$$

$$= \frac{0.037\text{v}_\text{m}^{1.8}x^{0.8}\rho y}{\nu^{-0.2}} = \frac{0.037\times50^{1.8}\times100^{1.8}\times1.2\times8.3}{\left(1.5\times10^{-5}\right)^{-0.2}} = 1819\text{N}$$

3. (i) (b) Apply Laplace formula:

$$\Delta p = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = \sigma\left(\frac{1}{\infty}+\frac{2}{0.05\text{mm}}\right) = h\rho_4 g$$

$$\sigma = 10.0\text{mm}\times0.025\text{mm}\times145\text{kgm}^{-3}\times9.8\text{ms}^{-2} = 355\,\mu\text{J m}^{-2}$$

(ii) $F = \rho w\left[h_2(u_2+c)^2 - h_1(u_1+c)^2\right]$, where w is the width of the channel. F comes from pressure in liquid over ends of the control volume: $w\int_0^{h_x}\rho g h\,dh = w\rho g\dfrac{h_x^2}{2} \rightarrow F = \dfrac{w\rho g}{2}\left(h_1^2 - h_2^2\right)$

By continuity $(u_2+c) = \dfrac{h_1}{h2}(u_1+c)$

$$\frac{w\rho g}{2}\left(h_1^2-h_2^2\right) = \rho w\left[h_2\frac{h_1^2}{h_2^2} - h_1\right](u_1+c)^2 \quad\Rightarrow\quad u_1 + c = \sqrt{gh_2}\left[\frac{h_1+h_2}{2h_1}\right]^{\frac{1}{2}}$$

4. Considering the pressure balance in the two limbs of the manometer:

$\Delta p = xg(\rho_m - \rho_w) = 0.05\times9.81\times10^3(13.6-1) = 6180\text{ Nm}^{-2}$, where $x = 50$ mm.

$$\text{v} = \sqrt{\frac{2\Delta p}{\rho_w}} = \sqrt{\frac{2\times6180}{10^3}} = 3.5\text{ ms}^{-1}$$

Hints and Tips for PHYM432, General Relativity and Cosmology

1. (i) This part of the question is mainly book work. The Lorentz transformation is

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx/c^2),$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Derived for case where frame S' moves along x-axis withs speed v relative to S and frames coincide at origin at time 0. For a light signal moving along the positive x axis, we have $x - ct = 0$ in S, and hence $x' - ct' = 0$ in S'. Now for arbitrary events, it follows from fact that transformation is linear, that $x' - ct' = A(x - ct) + D(x + ct)$ where A, D depend only on speed of the S' frame. But LHS vanishes for light signal moving along x axis, and hence $D = 0$. In a similar way, for light wave moving along $-x$, $x' + ct' = x + ct = 0$ and for arbitrary events $x' + ct' = B(x + ct)$
where B just depends on relative speed of frames v.

Now consider the observation of events from the view point of S', who sees S moving along $-x$ with speed v. Then $x + ct = C(x' + ct')$. However, since space is isotropic, this is same transformation as when S' moves along x with speed v. Hence C must be the same as A.

Thus $(x + ct) = A(x' + ct') = AB(x + ct)$ and $AB = 1$. Now to get A and B, consider location of origin of S' after time t. In frame S this is (ct, vt) and in frame S' is $(ct', 0)$.

$$-ct' = A(v - c)t, \quad ct' = B(v + c)t$$

Hence $-A(v - c) = B(v + c) = (v + c)/A$ or $A = \gamma(1 + v/c)$ and $B = \gamma(1 - v/c)$ remembering $AB = 1$. Here $\gamma = 1/\sqrt{(1 - v^2/c^2)}$.

Let $\cosh\phi = \gamma$ then

$$\sinh\phi = \sqrt{\cosh^2\phi - 1} = \sqrt{\gamma^2 - 1} = \gamma v/c$$

Since γ is greater than 1, we can always find a solution for ϕ.

Hence

$$x' = x\cosh\phi - ct\sinh\phi, \quad ct' = ct\cosh\phi - x\sinh\phi$$

$$x' - ct' = (\cosh\phi + \sinh\phi)(x - ct) = e^{\phi}(x - ct),$$
$$x' + ct' = (\cosh\phi - \sinh\phi)(x + ct) = e^{-\phi}(x + ct)$$

These vanish for light signals and demonstrate that the effect of Lorentz transformation is to 'stretch' $x - ct$.

Now $dx^{1\prime} = \gamma(dx^1 - (v/c)dx^0)$, $dx^{0\prime} = \gamma(dx^0 - (v/c)dx^1)$

Hence, as $dx^1 = (w_x/c)dx^0$ and $dx^{1\prime} = (w'_x/c)dx^{0\prime}$ then $w'_x = \frac{(w_x-v)}{1-vw_x/c^2}$, $w'_y = \frac{w_y}{\gamma(1-vw_x/c^2)}$

Applying this we get

$$\tan\alpha' = \frac{w'_y}{w'_x} = -\frac{w_y}{\gamma v}$$

2. From composition of velocities,

$$u' = (u-v)/(1-vu/c^2)$$

Hence

$$\begin{aligned}\frac{du'}{dt} = \frac{du}{dt}/(1-vu/c^2) + \frac{du}{dt}(u-v)v/(c^2(1-vu/c^2)^2) \\ = \frac{du}{dt}\frac{(1-v^2/c^2)}{(1-uv/c^2)^2}\end{aligned}$$

If $v = u$ then $u' = 0$ and

$$\frac{du'}{dt} = \gamma^2\frac{du}{dt}$$

Now $dt' = \gamma(1-v^2/c^2)dt$ and hence

$$\frac{du'}{dt'} = \gamma^3\frac{du}{dt}$$

Now use

$$\begin{aligned}\frac{d\gamma u}{dt} = \frac{d(u\sqrt{1/(1-u^2/c^2)})}{dt} \\ = \gamma\dot{u} + \gamma^3\frac{u^2\dot{u}}{c^2} \\ = \gamma^3\frac{du}{dt}\end{aligned}$$

$$\frac{du'}{dt'} = \frac{du\gamma}{dt}$$

If LHS is constant $\equiv \alpha$, then, integrating gives $\alpha t = \gamma u +$ con or $\alpha t = \gamma u$ if particle starts from rest.

Write this as

$$u = \frac{\alpha t}{\sqrt{1+\alpha^2 t^2/c^2}}$$

and integrate a second time

$$x = (c^2/\alpha)\sqrt{1+\alpha^2 t^2/c^2} + \mathrm{con}$$

or $x = (c^2/\alpha)\sqrt{1+\alpha^2 t^2/c^2} - (c^2/\alpha)$ if particle starts from origin.

In Newtonian limit,

$$x = \frac{1}{2}\alpha t^2$$

as expected.

3. This is largely bookwork. The cosmic microwave background arises from photons created in fireball era which have become red-shifted and decoupled from matter through the expansion of the Universe.

   Photons absorbed by ions promoting electronic transitions lead to an equilibrium between the photons and plasma. The distribution of photons with frequency is then given by the Bose-Einstein formula:

$$n_k(t) = \frac{1}{e^{\hbar\omega_k/kT(t)} - 1}$$

   where $T(t)$ is the temperature of the plasma. At some time, the plasma condensed into neutral atoms and equilibrium between photons and matter is lost. The temperature of the photons no longer equals that of matter. Rather, the photon frequency, and hence temperature, became red-shifted with the expansion.

   In the absence of a plasma, for matter dominated era, photons are conserved and hence energy density varies as $n\hbar\omega \sim n/R$ because of the gravitational red-shift. But as total number of photons is conserved, $n \sim 1/R^3$, and hence energy density varies as $1/R^4$. However, for a black body, this is proportional to T^4 and hence $T \sim 1/R$.

   Early Universe composed of hot plasma where a strong coupling between photons and a plasma exists which maintains equilibrium. As R increases, the plasma, with density ρ, cooled and this leads to energy change of photon gas. Second Friedmann eqn gives

$$\frac{d(\rho c^2 R^3)}{dt} = -3R^2 p \frac{dR}{dt}$$

   Here

$$\rho c^2 = aT^4,\ p = E/3V = aT^4/3$$

   Hence $\frac{d(T^4R^3)}{dR} = -R^2T^4$, or RT is constant. Hence for times $t << t_r$, TR is also constant.

   Gamow' s estimate of the background temperature starts with binding energy of deuteron $\approx 10^5$eV or $10^9 K$ and equals the energy of photons released by reaction. If

$T = 10^9 K$, need proper density of protons and neutrons, $n \sim 10^{18}$ cm^{-3}, so that 50% fuse into deuterium.

Now ρ_g now is 10^{-28} kg m^{-3} or 10^{-7} protons per cm^3. Roughly 1 proton per m^3. For matter dominated Universe:

$$\begin{aligned} \frac{R(t)}{R_o} &= \Big(\frac{n_o}{n(t)}\Big)^{1/3} \\ &\approx \Big(\frac{10^{-7}}{10^{18}}\Big)^{1/3} \\ &\sim 10^{-8} \\ Hence\ T_o &\approx 10K \end{aligned}$$

4. Cosmological principle assumes Universe is spatially homogeneous and isotropic on a sufficiently large scale ($>> 10^8$ lt. yrs.) Cosmic standard coordinates are a convenient frame of reference exploiting the symmetry of the Universe. Place clock at every galaxy; measure time by local clock and hence $g_{00} = 1$. Space is divided by grid lines that are tied to galaxies. As time evolves, grid lines move with galaxies and hence spatial coordinates of galaxy x^i are constant. Measure proper distance to two stars , one along say x-axis and the other along another axis, say y, from observer at origin.

$$\begin{aligned} dl_1^2 &= -g(t,0)dx_1^2 \\ dl_2^2 &= -g(t,0)dy_2^2 \end{aligned}$$

At later time $t = t'$, re-measure:

$$\begin{aligned} dl_1'^2 &= -g(t',0)dx_1^2 \\ dl_2'^2 &= -g(t',0)dy_2^2 \end{aligned}$$

Look now at local expansion:

$$\frac{dl_1'}{dl_1}, \quad \frac{dl_2'}{dl_2}$$

This cannot depend on orientation otherwise expansion anisotropic; nor can it depend on our position as space homogeneous. Hence

$$g(t,x) = R^2(t)g(0,0)$$

R called cosmic scale factor. Choose grid lines to lie along r, θ, ϕ axes. Consider two dimensional spatial surface with $\theta = \pi/2$. Recall Gaussian curvature k:

$$k = \frac{1}{2rg^2(r)}\frac{\partial g(r)}{\partial r}$$

and following integration, gives

$$g(r) = \frac{1}{1 - kr^2}$$

where constant, curvature scalar, selected to give unity at origin.

Hence Robertson-Walker metric is

$$\begin{aligned} ds^2 = c^2dt^2 - R^2(t)\{&\frac{1}{1 - kr^2}dr^2 \\ &+r^2 \sin^2\theta \; d\phi^2 + r^2 d\theta^2\} \end{aligned}$$

Friedmann equations: Consider a classical particle in a gravitational field due to a continuous density of matter ρ. The energy of a galaxy of unit mass at a proper distance $rR(t)$ is $\frac{1}{2}r^2\dot{R}^2 - \frac{4\pi G}{3}\rho r^2 R^2 = -\frac{1}{2}kr^2$, or $\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -k$. It turns out that in GR, k can only be ± 1, or 0.

Consider the change in the energy of a region of the Universe by radiative emission. This is $\delta(\rho c^2 R^3)$ while the work done by radiation pressure on the region is $-p\delta R^3$. Hence $\frac{d\rho c^2 R^3}{dt} = -3R^2 p\frac{dR}{dt}$, or $\dot{\rho} = \frac{-3\dot{R}}{R}(p/c^2 + \rho)$.

These are the Friedmann equations. Solution requires equation of state $p = p(\rho)$. They lead to three types of Universe depending on whether $k > 0, = 0$ or < 0. These are closed or open and the expansion is cyclic or monotonic.

In early Universe, gravitation terms dominates k in Friedmann eqn. Hence choose $k = 0$.

$$\frac{d\rho c^2 R^3}{dt} = -3R^2 p\dot{R}, \quad p = \rho c^2/3$$

$$\dot{\rho}/\rho = -4\dot{R}/R \;\; or \;\; \rho R^4 = \rho_0 R_0^4 \;\; \text{.Now as } \rho c^2 = aT^4, \quad T = T_0\frac{R_0}{R}$$

$$\text{Friedmann first eqn} \;\; \dot{R}^2 = \frac{8\pi G\rho R^2}{3}, \quad R\dot{R} = \sqrt{8\pi G\rho_0 R_0^4/3}$$

$$\text{Hence} \;\; \frac{1}{2}R^2(t) = \sqrt{\frac{8\pi G\rho_0 R_0^4}{3}}t$$

PHYSICS EXAMINATION PROBLEMS
SOLUTIONS AND HINTS FOR STUDENT SELF-STUDY

Module Code	**PHYM434**
Name of module	**Signal Processing**
Date of examination	**May 2010**

1\. If $k = \sqrt{2}$, $|H| = \left(1 + \frac{\omega^4}{\omega_0^4}\right)^{-1/2}$

For compound filter: this is a 4th-order low-pass, with

$$|H|^2 = 1/\left[1 + (k_1^2-2)x + x^2\right]\left[1 + (k_2^2-2)x + x^2\right], \text{ where } x = \frac{\omega^2}{\omega_0^2}$$

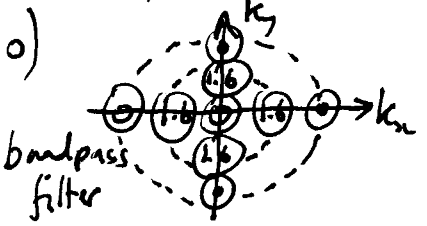
If $(k_1^2-2) + (k_2^2-2) = 0$ and $(k_1^2-2)(k_2^2-2) = -2$, gives $|H|^2 = \left(1 + \frac{\omega^8}{\omega_0^8}\right)^{-1/2}$

Solns are $k_1^2 - 2 = \sqrt{2}$, $k_2^2 - 2 = -\sqrt{2}$ (or vice versa)

2\. (a) all pixels zero (b) columns −40 0 40 0 −40 0 (c) all pixels zero

$|H| = 0$ at $(k_x, k_y) = (0,0), \left(\frac{\pi}{5}, 0\right), \left(-\frac{\pi}{5}, 0\right)$

$|H| = 1{\cdot}6$ at $(k_x, k_y) = \left(\frac{\pi}{25}, 0\right), \left(-\frac{\pi}{25}, 0\right)$



3\. $PSF = \frac{1}{r}$, $MTF = \frac{1}{k} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$

2D filter $H(k_x, k_y) = k = \sqrt{k_x^2 + k_y^2}$

For 1D filter,

$$\begin{tikzcd} p(s,\phi) \arrow[r, "F.T."] & P(k,\phi) \arrow[r, "filter"] & {|k|P(k,\phi)} \arrow[d, "I.F.T."] \\ & & q(s,\phi) \end{tikzcd}$$

4\.

(i) $\dfrac{e^{-T_E/T_2^*}}{e^{-T_E/T_2}} = 0{\cdot}25 \Rightarrow \dfrac{1}{T_2^*} - \dfrac{1}{T_2} = \dfrac{\ln 4}{T_E}$

$$\Rightarrow \frac{\gamma \Delta B_z}{2} = \frac{\ln 4}{T_E} \Rightarrow \Delta B_z = 6{\cdot}9 \times 10^{-7}\,T$$

(ii) b/w is 800Hz. Maximum slice thickness is 12mm.

800Hz equivalent to 18·78 μT

i.e. gradient $= \frac{18{\cdot}78}{12} \times 10^{-3} = 1{\cdot}56$ mT m^{-1}