PHY2201 Summary Sheet 20

As $T \to 0$, fermions will fill all available states up to some maximum energy E_F or equivalently a max. momentum p_F, the Fermi momentum.

Hence number of states contained within an octant of momentum space, radius $p_F = N/2$ (because each translational momentum state actually comprises TWO distinct quantum states, with the electron spin "up" and spin "down" respectively).

$$\frac{1}{8}\frac{4\pi}{3}p_F^3 \Big/ \left(\frac{\hbar\pi}{L}\right)^3 = \frac{N}{2}$$

$$\therefore\ p_F^3 = \frac{3}{8\pi}\frac{N}{V}h^3$$

writing $N/V = n$, the particle number density

$$\boxed{E_F = \frac{p_F^2}{2m} = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3}}$$

e.g. for the conduction electrons in a metal

$$E_F = \frac{\left(6.6\times10^{-34}\right)^2}{2\times10^{-30}}\left(\frac{3\times10^{28}}{8\pi}\right)^{2/3} \sim 1.5\text{eV}$$

Fermi-Dirac distribution.

Equilibrium distribution of energy U over N particles when

a) only ONE particle per state is allowed (c.f. Boltzmann distribution, any number of particles per state were allowed).
b) the particles are indistinguishable (cf Boltzmann distribution, the particles were distinguishable).

Quantum states form a densely-spaced near-continuum. Divide these states into "bands" of nearly identical energy. Hence band i has a characteristic energy E_i, number of states ω_i and holds n_i particles.

Total number of microstates Ω_{total} is given by

$$\Omega_{\text{total}} = \prod_{i=1}^{\text{all bands}} \Omega_i$$

Where Ω_i = the total number of ways to choose n_i indistinguishable objects from ω_i possibilities (c.f. coin-flipping) hence

$$\Omega_{\text{total}} = \prod_{i=1}^{\text{all bands}} \frac{\omega_i!}{(\omega_i - n_i)!\,n_i!}$$

As with the Boltzmann distribution, we obtain the equilibrium distribution by seeking the n_i 's that maximise $\ln \Omega_{\text{total}}$ subject to the constraints $\sum_{i=1}^{\text{all bands}} n_i = N \quad \sum_{i=1}^{\text{all bands}} n_i E_i = U$

Solution (see supplement sheet 4)

$$\boxed{\frac{n_i}{\omega_i} = \frac{1}{\exp\left((E_i - E_F)/k_{\text{B}}T\right) + 1}}$$

Fermi-Dirac distribution

With E_F is a constant (the Fermi Energy).