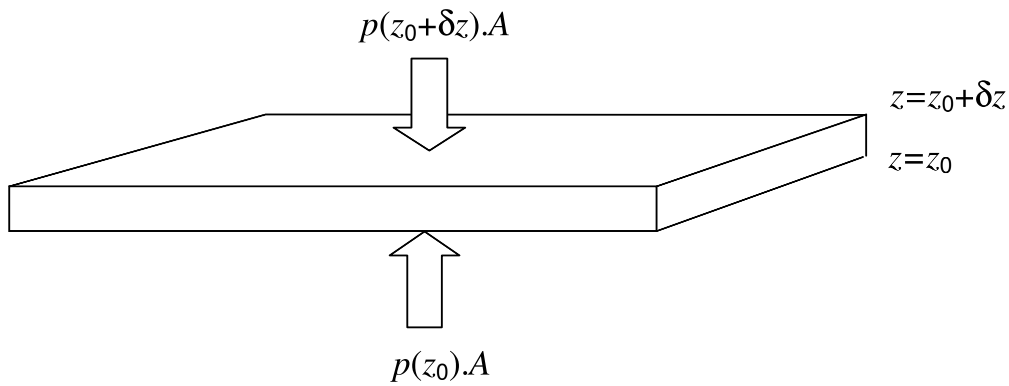
PHY2201 Summary Sheet 6

Example of the Boltzmann factor:

Consider a mass of isothermal ideal gas, at temperature T. For a thin slab of gas at height z_0, thickness δz and cross-sectional area A to not fall under gravity requires :



Hydrostatic equilibrium:-

$$p(z_0)A - p(z_0+\delta z)A = \rho(z_0)\,A\,\delta z\,g$$
$$\therefore\ p(z_0) - p(z_0+\delta z) = \rho(z_0)\ \delta z\ g$$

Expanding $p(z)$ in a Taylor series:

$$p(z_0+\delta z) = p(z_0) + \left.\frac{\mathrm{d}p(z)}{\mathrm{d}z}\right|_{z_0}\delta z + \frac{1}{2!}\left.\frac{\mathrm{d}^2p(z)}{\mathrm{d}z^2}\right|_{z_0}\delta z^2 + \frac{1}{3!}\left.\frac{\mathrm{d}^3p(z)}{\mathrm{d}z^3}\right|_{z_0}\delta z^3 + ...\mathrm{O}(\delta z^4)$$

In the limit that the slab thickness $\delta z \to 0$,

$$p(z_0+\mathrm{d}z) - p(z_0) = \left.\frac{\mathrm{d}p(z)}{\mathrm{d}z}\right|_{z_0}\mathrm{d}z\ ,$$

Hence,

$$\rho(z_0)g\ \mathrm{d}z = -\left.\frac{\mathrm{d}p(z)}{\mathrm{d}z}\right|_{z_0}\mathrm{d}z\ ,$$

Hence

$$\rho(z)g = -\frac{\mathrm{d}p(z)}{\mathrm{d}z}$$

$$\therefore \frac{\mathrm{d}p(z)}{\mathrm{d}z} = -n(z)m_A g$$

With m_A being the mass of one gas atom and n being the number density of gas atoms.

Ideal gas equation of state $pV = Nk_BT \Rightarrow p = \frac{N}{V}k_BT = nk_BT$, hence

$$k_BT\frac{\mathrm{d}n(z)}{\mathrm{d}z} = -n(z)m_A g \Rightarrow \frac{\mathrm{d}n(z)}{\mathrm{d}z} = -\frac{m_A g}{k_BT}n(z)$$

$$\therefore n(z) = n(z=0).\exp\left(-\frac{m_A g}{k_BT}z\right)$$

Hence $n(z)$, $\rho(z)$ and $p(z)$ all fall exponentially with height.

$m_A g\, z$ is the <u>gravitational potential energy</u> of a gas atom at height z. Since $n(z) \propto$ probability of finding a gas atom at height z, suggests that the probability of finding a gas atom in an "energy level" of value $\varepsilon(z)$ is proportional to

$$\boxed{\exp\left(-\frac{\varepsilon(z)}{k_BT}\right)}$$

Boltzmann factor

This factor is of universal validity; whenever an ensemble of classical particles are in equilibrium at temperature T, the probability of an energy level of value $\varepsilon(z)$ being "occupied" by a particle of the ensemble varies as the exponential of $-\varepsilon(z)/k_BT$.