PY2201 Summary Sheet 4

To relate the Maxwell velocity, speed and energy distributions

Maxwell velocity distribution:

$$p(v_x) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{m}{k_B T}} \cdot \exp\left(-\frac{mv_x^2}{2k_B T}\right) \quad \text{ditto } v_y,\ v_z$$

Hence probability of finding a particle to possess velocity in range (v_x, v_y, v_z) to (v_x+dv_x , v_y+dv_y ,v_z +dv_z) is

$$p(v_x)p(v_x)p(v_x)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z =$$

$$\frac{1}{(2\pi)^{3/2}} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot \exp\left(-\frac{mv_x^2}{2k_B T}\right)\exp\left(-\frac{mv_y^2}{2k_B T}\right)\exp\left(-\frac{mv_z^2}{2k_B T}\right)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$

$$= \frac{1}{(2\pi)^{3/2}} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot \exp\left(-\frac{m\left(v_x^2 + v_y^2 + v_z^2\right)}{2k_B T}\right)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$

$$= \frac{1}{(2\pi)^{3/2}} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$

To obtain the probability of finding a particle to possess a speed in the range v to v+dv, multiply this by the "number" of velocity states that contribute to this speed range i.e. $4\pi v^2\mathrm{d}v/(\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z)$:-

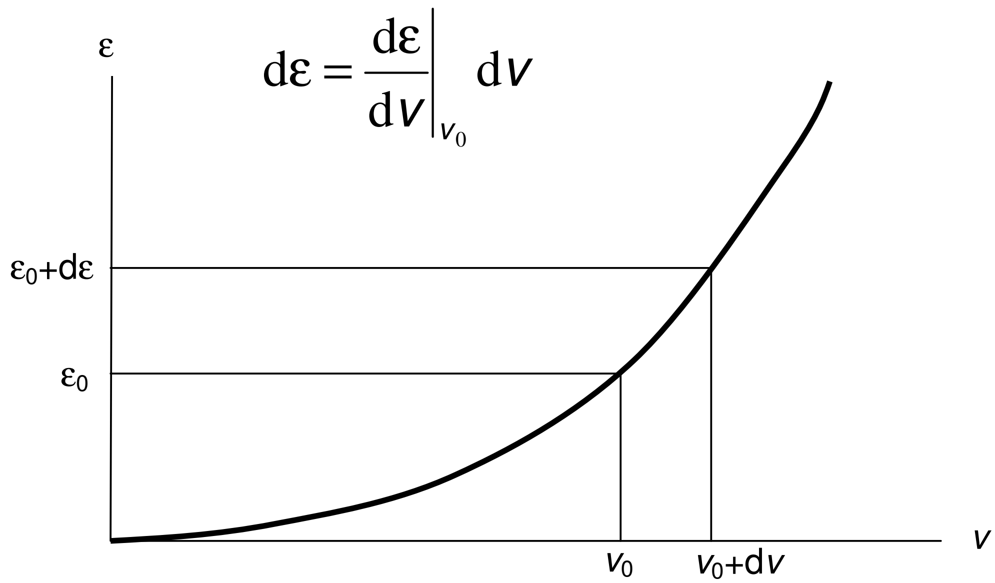
$$p(v)\mathrm{d}v = \frac{1}{(2\pi)^{3/2}} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2k_B T}\right)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z \cdot \frac{4\pi v^2\mathrm{d}v}{\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot v^2 \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot \mathrm{d}v$$

Noting that the kinetic energy of a particle, ε is given by

$$\varepsilon = \frac{mv^2}{2},$$

we can exploit the 1:1 correspondence between ε and v to reformulate the speed distribution as a kinetic energy distribution :-



Since a speed between v_0 and v_0+dv implies an energy between ε_0 and ε_0+dε, with $d\varepsilon = \frac{d\varepsilon}{dv} dv$,
the probability of obtaining a speed between v_0 and v_0+dv equals probability of obtaining an energy between ε_0 and ε_0+dε.

hence, with $\varepsilon = \frac{mv^2}{2}; d\varepsilon = m\,v\,dv = \sqrt{2m\varepsilon}\,dv$

$$p(v)dv = p(\varepsilon)d\varepsilon = \left(\frac{2}{\pi}\right)^{1/2} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot v^2 \cdot \exp\left(-\frac{mv^2}{2k_B T}\right) \cdot dv$$

$$\therefore\ p(\varepsilon)d\varepsilon = \left(\frac{2}{\pi}\right)^{1/2} \cdot \left(\frac{m}{k_B T}\right)^{3/2} \cdot \frac{2\varepsilon}{m} \cdot \exp\left(-\frac{\varepsilon}{k_B T}\right) \cdot \frac{d\varepsilon}{\sqrt{2m\varepsilon}}$$

$$= \frac{2}{(\pi)^{1/2}} \cdot \left(\frac{1}{k_B T}\right)^{3/2} \cdot \varepsilon^{1/2} \cdot \exp\left(-\frac{\varepsilon}{k_B T}\right) \cdot d\varepsilon$$