Calculus of Many Variables

Bibliography

The course text:

1. "Advanced Mathematics for Engineers and Scientists" Spiegel, Schaum Outline Series.

Alternatives:

2. "Mathematical Methods for Science Students", G. Stephenson, Longman.
3. “The Chemistry Maths Book”, Erich Steiner, O. U. P.

More advanced:

4. “Mathematical Methods for Physicists”, Arfken, Academic Press.

More formal:

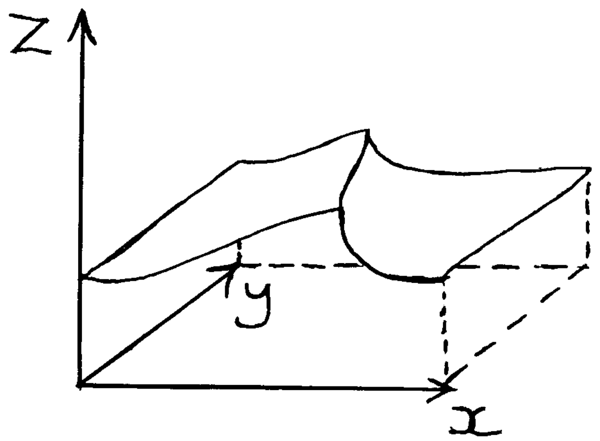
5. "Differential Calculus", P. J. Hilton, Library of Mathematics , ed. Walter Ledermann, Routledge & Kegan Paul (RKP)
6. "Partial Derivatives", P. J. Hilton, RKP.
7. "Multiple Integrals", W. Ledermann, RKP.

Motivation

In physics many quantities of interest depend upon more than one variable.

Example: the height of a wave z on the ocean depends upon the coordinates x and y, and also upon the time t. We write

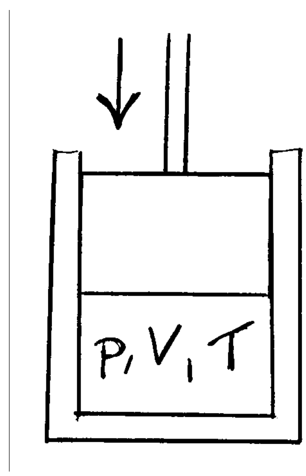
$z(x,y,t)$ i.e. z is a function of x, y and t.



Example: The ideal gas law states that

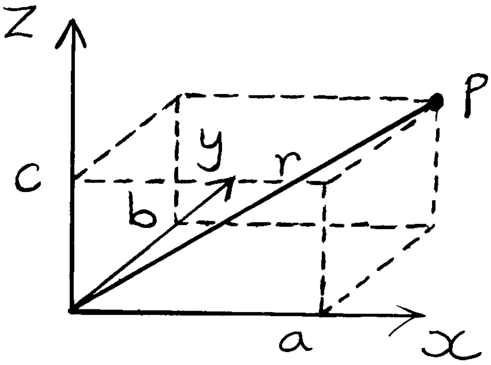
$$p(T,V) = nkT\ /\ V$$

p is a function of T and V.



<u>Coordinate Systems</u>

Cartesian coordinates (x, y, z) label a point P in 3 dimensional space.

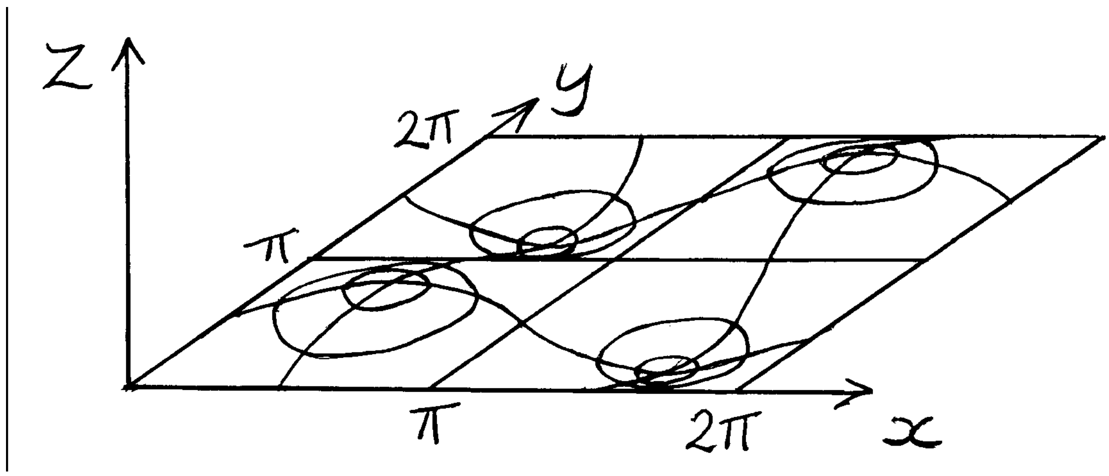


The distance of P from the origin

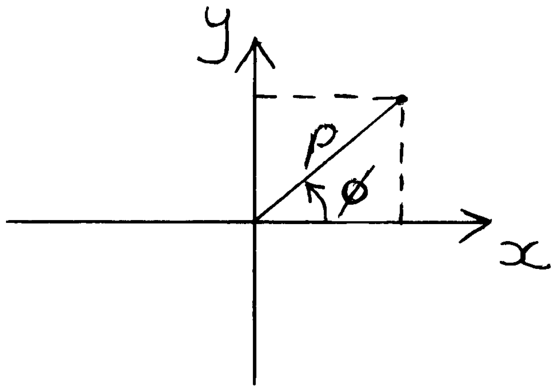
$$r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}},$$

is a function of x, y and z.
Other functions of (x, y, z) are harder to visualise. The value of a function of two variables, such as x and y can be plotted along the z axis. For example

$$f(x, y) = \sin x \sin y$$



Polar Coordinates in 2 Dimensions



x and y are functions of ρ and ϕ.

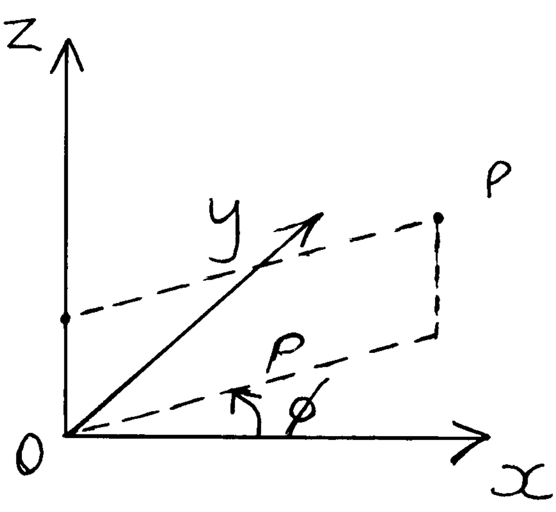
$$x = \rho\cos\phi, \;\; y = \rho\sin\phi.$$

We can also write ρ and ϕ as functions of x and y,

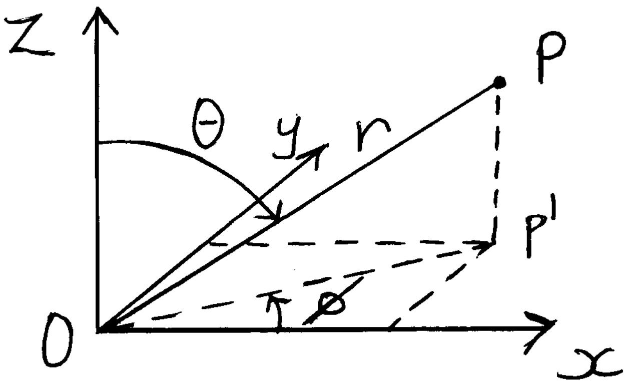
$$\rho = \left(x^2 + y^2\right)^{\frac{1}{2}}, \;\; \phi = \tan^{-1}\frac{y}{x}.$$

Cylindrical Polar Coordinates

In 3D we use (ρ, ϕ, z)



Spherical Polar Coordinates



(x, y, z) are functions of (r, θ, ϕ)

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

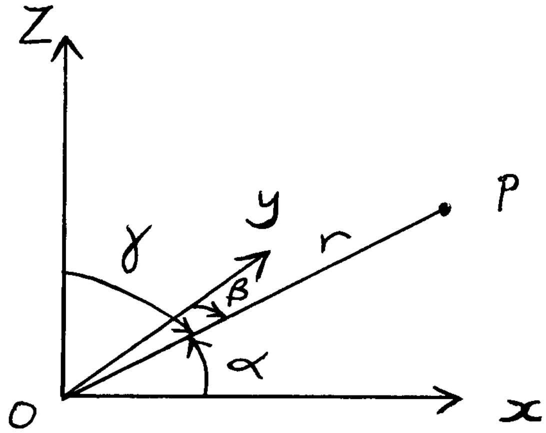
Or we could write (r, θ, ϕ) as functions of (x, y, z)

$$r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}},$$
$$\phi = \tan^{-1}\frac{y}{x},$$
$$\theta = \cos^{-1}\left(\frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}\right)$$

In general we will choose the coordinate system that is most convenient for a given problem.

Direction Cosines

One other way of labelling the position in 3 dimensions:



r, α, β, γ are used to specify the position.

The cosines of the angles α, β, γ are known as *the direction cosines*.

Differentiation

The derivative of $y = f(x)$ at a point x is the slope of the tangent to the curve at that point.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

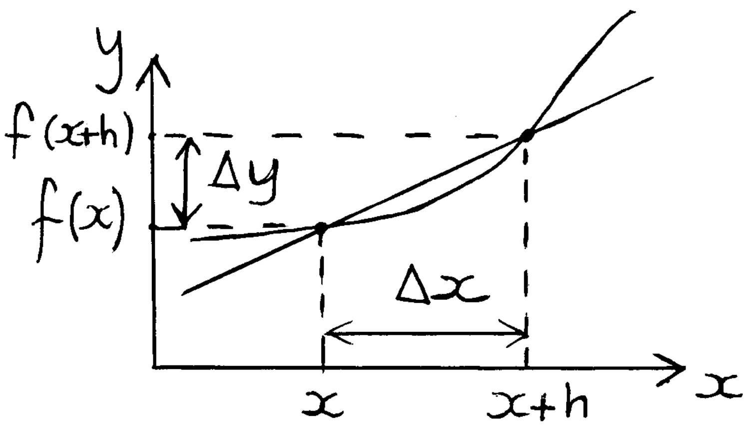
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

where

$$\Delta x = h,$$

$$\Delta y = f(x + \Delta x) - f(x) = f(x+h) - f(x)$$



Notice that

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

The *differential* dy is given by

$$dy = \frac{dy}{dx}dx = f'(x)dx$$

Example: Show that the derivative of $y = x^n$ is nx^{n-1}

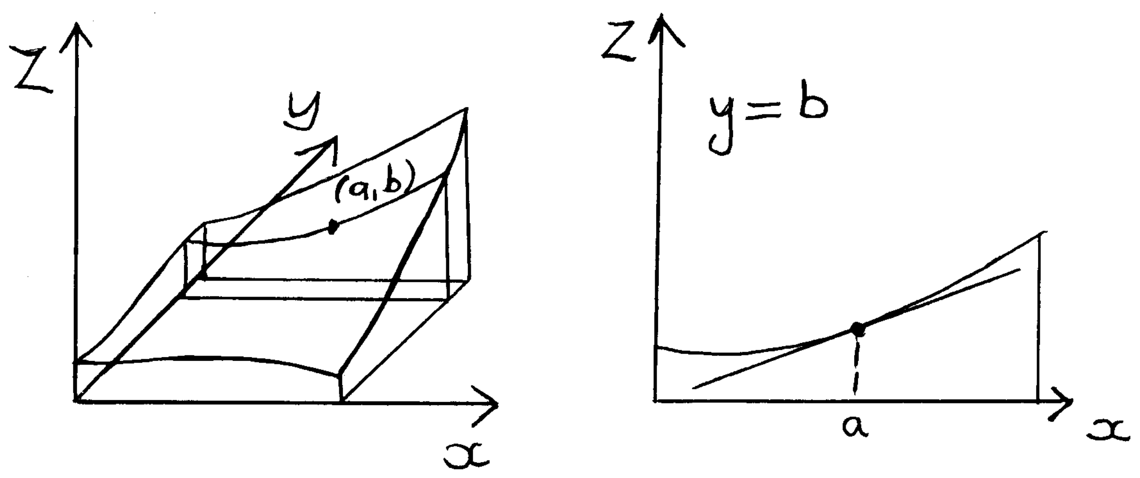
Partial Differentiation

The *partial* derivatives of $f(x,y)$ with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$
$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}.$$

These are tangents to cross-sections through the surface $z = f(x,y)$.

Consider $\dfrac{\partial f}{\partial x}$ at $x = a$ when $y = b$:



Notation

∂ is used instead of d in partial derivatives

$\dfrac{\partial f}{\partial x}$ may also be written as f_x or $\left(\dfrac{\partial f}{\partial x}\right)_y$.

In the latter case the subscript y indicates that y is held constant while $f(x, y)$ is differentiated with respect to x.

Example: Find the partial derivatives with respect to x and y of

$$f(x, y) = xy^2 + 3x + 2y + 4y \sin x$$

Warning!

We <u>must</u> pay attention to which variables are being kept constant.

Example: If $x = \rho\cos\phi$ and $y = \rho\sin\phi$, then does $\dfrac{\partial\rho}{\partial x} = \dfrac{1}{\dfrac{\partial x}{\partial\rho}}$?

The Differential

Consider how the value of $f(x,y)$ changes on moving from (x,y) to $(x+\Delta x, y+\Delta y)$. The change in f may be written as

$$\begin{aligned}
\Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) \\
&\quad + f(x, y+\Delta y) - f(x,y) \\
&= \Delta x\left[\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right] \\
&\quad + \Delta y\left[\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right]
\end{aligned}$$

for small Δx, Δy,

$$\approx \frac{\partial f(x, y+\Delta y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

$$= \left[\frac{\partial f(x,y)}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)\Delta y\right]\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

$$\approx \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

So the *differential* may be written as

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

The Reciprocal and Reciprocity Theorems

Suppose that x, y, z are related to each other so that there are only 2 independent variables e.g. a surface in 3D space. We could write $z(x, y)$ or $x(y, z)$ and then

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz$$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy \ .$$

Substituting one into the other,

$$dx = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y dx$$

$$+ \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] dy$$

But if x and y are the independent variables then the coefficients of dx and dy must be equal to zero, and so from the coefficient of dx we obtain:

$$\left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{\left(\frac{\partial z}{\partial x}\right)_y}$$

(The Reciprocal Theorem)

Applying the Reciprocal Theorem to the coefficient of *dy* we obtain:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

(The Reciprocity Theorem)

Partial and Total Derivatives

If x and y are functions of a parameter t then $f(x, y)$ is *implicitly* a function of the single parameter t. What is the rate of change of $f(x, y)$ with t?

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right]$$

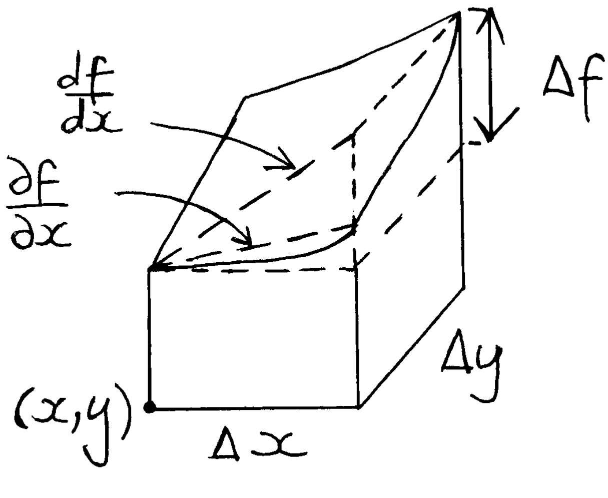
so $$\frac{df}{dt} = \lim_{\Delta t \to 0} \left[\frac{\partial f}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta t} \right]$$

$$= \left(\frac{\partial f}{\partial x} \right)_y \frac{dx}{dt} + \left(\frac{\partial f}{\partial y} \right)_x \frac{dy}{dt}$$

$\dfrac{df}{dt}$ is the *total* derivative of f w.r.t. t.

If x is in fact the parameter then

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial x} \right)_y + \left(\frac{\partial f}{\partial y} \right)_x \frac{dy}{dx}$$



For a function $f(x, y, z)$ where z is a function of x and y we can write

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_y .$$

Notice that:

(i) we must use the subscripts

(ii) there are now no total derivatives

Differentiating implicit functions

Let y be an implicit function of x. For example, consider

$$xy = \sin(x+y).$$

How do we find $\frac{dy}{dx}$? Let us write

$$f(x,y) = xy - \sin(x+y) = 0$$

Then

$$\frac{df}{dx} = 0 = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x} = -\frac{f_x}{f_y}$$

which for our example gives

$$\frac{dy}{dx} = -\frac{y - \cos(x+y)}{x - \cos(x+y)}$$

Higher order partial derivatives

Second and higher order partial derivatives may be defined simply by differentiating the first derivatives again, so that

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)_y \right)_y = f_{xx} ,$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)_x \right)_x = f_{yy} ,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)_x \right)_y = f_{xy} ,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)_y \right)_x = f_{yx} .$$

The variables being held constant can be written as subscripts but this soon becomes cumbersome and often they are omitted.

For most functions that we will meet it can be shown that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

in which case the differential

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

is *exact* i.e. the change in the value of f is independent of the path taken when moving between two points with different coordinates.

The work done by a force may be written as

$$dW = \mathbf{F} \cdot \mathbf{dr} = F_x dx + F_y dy$$

where the subscripts now refer to the component of the vector $\mathbf{F}$.

This differential would be exact if (x, y) were the coordinates of a charge in an electric field. The same would not be true if (x, y) were the coordinates of a ball bearing in a jar of treacle. Forces that lead to exact differentials are said to be *conservative*.

The second derivative of an implicit function

How do we calculate $\dfrac{d^2y}{dx^2}$ when y is an implicit function of x?

Again we write an equation of the form

$$f(x,y)=0.$$

Then by repeated application of the *operator* used to calculate the first derivative

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\left(\left(\frac{\partial}{\partial x}\right)_y+\left(\frac{\partial}{\partial y}\right)_x\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

and substituting the previous expression that we found before for $\dfrac{dy}{dx}$,

$$\frac{d^2y}{dx^2}=\left(\left(\frac{\partial}{\partial x}\right)_y-\frac{f_x}{f_y}\left(\frac{\partial}{\partial y}\right)_x\right)\left(-\frac{f_x}{f_y}\right)$$

$$\frac{d^2y}{dx^2}=-\frac{f_{xx}}{f_y}+\frac{f_x f_{xy}}{f_y{}^2}-\frac{f_x}{f_y}\left(-\frac{f_{yx}}{f_y}+\frac{f_x f_{yy}}{f_y{}^2}\right)$$

$$\frac{d^2y}{dx^2}=-\frac{f_x^2 f_{yy}-2f_x f_y f_{xy}+f_y^2 f_{xx}}{f_y^3}$$

Example: compute $\frac{d^2y}{dx^2}$ for the case that $x^2 + y^2 = a^2$

The Chain Rule

If f is a function of a variable u and u is a function x then we may differentiate f with respect to x by the Chain Rule

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

There is also a Chain Rule for functions of more than one variable.

If f is a function of (u,v) which are functions of (x,y) then

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y,$$

$$\left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial y}\right)_x + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

note which variables are being kept constant!

Example: If $f = xy$ calculate the partial derivatives with respect to the polar coordinates ρ and ϕ, by the chain rule and explicitly by substitution.

We will often be concerned with the form of *differential operators* in different coordinate systems.

Example: Transform the operator $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$ into polar coordinates.