Exercises 1

1. Verify that the expressions

$$r = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}},$$

$$\phi = \tan^{-1}\frac{y}{x},$$

$$\theta = \cos^{-1}\left(\frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}\right)$$

follow from the definitions

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta.$$

2. A point has cylindrical polar coordinates $(\rho,\phi,z)=\left(3,\frac{\pi}{4},6\right)$. Find the Cartesian and spherical polar coordinates of the point.

3. Using the definition of the derivative given in the lecture, prove the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Hint: make a Taylor expansion of $u(x+h)$ and $v(x+h)$, and then retain only first order terms.

4. Differentiate the following expressions with respect to x:

(a) $y = \sin\left[(2x+3)^2\right]$

(b) $y = \sin^{-1}(x)$

Exercises 2

1. Find the partial derivatives with respect to x and y of the functions

   (a) $f(x,y) = 5x^2y^3$.

   (b) $f(x,y) = e^{ax}\sin(xy)$

2. Find the partial derivatives with respect to x, y and z of the function

$$f(x,y,z) = 2x^2yz^3 + y^2z.$$

3. Find the rate of change with time t of the function

$$f(\rho,\phi) = \rho\cos\phi$$

   given that $\rho = at^2$, and $\phi = bt$.

4. Consider the function

$$f(x,y) = x^2 + y^2.$$

   Calculate the partial and total derivatives of f with respect to x given that $y = x^2$. Make a sketch of $f(x,y)$ and $y(x)$ and show the geometrical significance of the two derivatives.

Exercises 3

1. If $y = \exp(x+y)$, then use the formula given for differentiation of an implicit function to find $\dfrac{dy}{dx}$, and express your answer in terms of y. Then solve the original equation to obtain x in terms of y and obtain $\dfrac{dy}{dx}$ by explicit differentiation.

2. If

$$f(x,y) = x^2 y + \sin x + \exp(x+y)$$

then calculate $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$. Show that the last two second derivatives are identical.

3. If $y = \exp(x+y)$, then find $\dfrac{d^2 y}{dx^2}$ using the expression given in the lecture for differentiation of an implicit function. Then check your answer by explicitly differentiating the result of question 1.

4. $f(u,v) = u^2 \ln v$, where $u = x + y$ and $v = x - y$.

(a) Use the Chain Rule to calculate $\left(\dfrac{\partial f}{\partial x}\right)_y$ and $\left(\dfrac{\partial f}{\partial y}\right)_x$, expressing your answer in terms of x and y.

(b) Express f in terms of x and y and directly calculate $\left(\dfrac{\partial f}{\partial x}\right)_y$ and $\left(\dfrac{\partial f}{\partial y}\right)_x$. Confirm that these are equivalent to the expressions obtained in part (a).