**Example**: evaluate the integral

$$I = \int_0^a \int_0^b x\, e^{\beta y} dy dx$$

Answer: first perform the integral w.r.t. $y$ while holding $x$ constant

$$I = \int_0^a \left[ \frac{x}{\beta} e^{\beta y} \right]_0^b dx = \int_0^a \left[ \frac{x}{\beta} \left( e^{\beta b} - 1 \right) \right] dx$$

$$I = \left[ \frac{x^2}{2\beta} \left( e^{\beta b} - 1 \right) \right]_0^a = \frac{a^2}{2\beta} \left( e^{\beta b} - 1 \right)$$

**Example**: A disc of radius $a$ centred at the origin has mass per unit area

$$m = c\sqrt{a^2 - x^2 - y^2}\,.$$

Find the total mass of the disc.

Answer: The mass is given by

$$M = \iint m\, dA$$

but the density can be written in the form

$$m = c\sqrt{a^2 - \rho^2}$$

and the limits may be more easily written in polar coordinates. We therefore choose to evaluate the mass in polar coordinates

$$M = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} c\sqrt{a^2 - \rho^2}\,(\rho d\rho d\phi)$$

which we may rewrite in the form

$$M = c\int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{a} \rho\sqrt{a^2 - \rho^2}\, d\rho$$

$$M = c[\phi]_0^{2\pi}\left[-\frac{1}{3}\left(a^2 - \rho^2\right)^{\frac{3}{2}}\right]_0^a$$

$$M = 2\pi c\left[+\frac{1}{3}\left(a^2\right)^{\frac{3}{2}}\right] = \frac{2}{3}\pi c a^3$$

**Example**: Calculate the area of the *astroid* which is defined by

$$x = a\cos^3 t,\ y = a\sin^3 t \text{ where } 0 \le t \le 2\pi.$$

Answer: We will use a trick. Green's Theorem lets us write the area as

$$A = \iint dx\,dy = \tfrac{1}{2}\oint (x\,dy - y\,dx).$$

The line integral is then evaluated by parameterisation

$$A = \frac{1}{2}\int_0^{2\pi}\left( x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi}\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)dt$$

$$A = \frac{3a^2}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt = \frac{3a^2}{8}\int_0^{2\pi}\sin^2 2t\,dt$$

$$A = \frac{3a^2}{16}\int_0^{2\pi}(1-\cos 4t)\,dt = \frac{3a^2}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^{2\pi} = \frac{3a^2}{8}\pi$$

**Example**: we will recalculate the integral from the previous problem (see page 29 of Lecture Notes) using spherical polar coordinates.

$$I = \iint_S x^2 dS = \iint_S (a\cos\phi\sin\theta)^2 a^2 \sin\theta d\theta d\phi$$

$$= a^4 \int_0^{2\pi} \cos^2\phi d\phi \int_0^{\pi/2} \sin^3\theta d\theta$$

$$I = a^4 \int_0^{2\pi} \frac{1}{2}(1+\cos 2\phi) d\phi \int_0^{\pi/2} \sin\theta (1-\cos^2\theta) d\theta$$

$$I = a^4 \left[\frac{\phi}{2}\right]_0^{2\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2\pi}{3} a^4$$

**Example**: Evaluate $I = \iint_S x^2 dS$ where $S$ is now the cylindrical surface defined by $\rho = a$, $-b \le z \le +b$.

$$I = \iint_S x^2 dS = \iint_S (a\cos\phi)^2 \, a \, d\phi \, dz = a^3 \int_0^{2\pi} \cos^2\phi \int_{-b}^{+b} dz$$

$$I = a^3 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\phi) d\phi \int_{-b}^{+b} dz$$

$$= \frac{a^3}{2}\left[\phi + \frac{1}{2}\sin 2\phi\right]_0^{2\pi} [z]_{-b}^{+b} = 2\pi a^3 b$$

**Example**: calculate the integral $I = \iint\limits_{\Re} xy\, dA$ where the region $\Re$ is the quarter circle in the first quadrant with radius $a$.

Answer: the integrand and the limits of the integral are easily written in polar coordinates and an expression for $dA$ was obtained previously, therefore

$$I = \int_0^{\pi/2} \int_0^a (\rho \cos\phi)(\rho \sin\phi)\rho \, d\rho \, d\phi$$

$$I = \int_0^{\pi/2} \left( \frac{1}{2} \sin 2\phi \right) d\phi \int_0^a \rho^3 d\rho$$

$$I = \left[ -\frac{1}{4} \cos 2\phi \right]_0^{\pi/2} \left[ \frac{1}{4} \rho^4 \right]_0^a = \frac{a^4}{8}$$

The result is the same as that obtained in the exercise earlier in the lecture using Cartesian coordinates. Notice how the limits are simpler in polar coordinates.

**Example**: Evaluate the integral

$$I(a) = \int_{-\infty}^{+\infty} \exp\left(-ax^2\right) dx$$

We can use a trick by writing

$$[I(a)]^2 = \left( \int_{-\infty}^{+\infty} \exp\left(-ax^2\right) dx \right)\left( \int_{-\infty}^{+\infty} \exp\left(-ay^2\right) dy \right)$$

$$[I(a)]^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-a\left(x^2 + y^2\right)\right) dx\, dy.$$

changing to polar coordinates

$$[I(a)]^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-a\rho^2\right) \rho d\rho\, d\phi$$

$$[I(a)]^2 = \int_0^{\infty} \exp\left(-a\rho^2\right) \rho d\rho \int_0^{2\pi} d\phi$$

$$[I(a)]^2 = \left[ -\frac{1}{2a} \exp\left(-a\rho^2\right) \right]_0^{\infty} [\phi]_0^{2\pi}$$

$$[I(a)]^2 = \frac{1}{2a} 2\pi$$

$$I(a) = \left( \frac{\pi}{a} \right)^{\frac{1}{2}}$$