PROBLEMS

1 Functions of a Complex Variable

1.1 If $z_1 = 3 + 2i$ and $z_2 = -1 + i$, find $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and z_1 / z_2. Evaluate $z_1^* z_1$, $z_2^* z_2$ and $z_1^* z_2$.

1.2 Find the real and imaginary parts of

(a) $(2 + 3i)/(3 + 2i)$,

(b) $\ln\{(\sqrt{3} + i)/2\}$,

(c) $(1 + i)^{iy}$,

(d) $1/i^5$,

(e) $(-1/2 + i\sqrt{3}/2)^2$.

1.3 Express each of the following functions in the form $f(z) = u(x, y) + iv(x, y)$, where u and v are real:

(a) $f(z) = z^3$,
(b) $f(z) = \exp(-z)$,
(c) $f(z) = 1/(1 - z)$,
(d) $f(z) = \ln z$,
(e) $f(z) = \tan z$.

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1.4 Use Cauchy-Riemann equations to find out which of the following functions are analytic:

(a) $f(z) = z^2$,

(b) $f(z) = \exp(3z)$,

(c) $f(z) = \left(z^*\right)^2$,

(d) $f(z) = \left|z^2\right|$,

(e) $f(z) = \cos(z)$.

1.5 Find the constant λ, such that the function $u(x, y) = \exp(\lambda x)\cos y$ is the real part of the analytic function $f(z) = u(x, y) + iv(x, y)$. Find the corresponding imaginary part v.

1.6 Evaluate $\int_0^{1+i} z^2 dz$

(a) along the parabola $x = t$, $y = t^2$ where $0 \le t \le 1$,

(b) along the straight line joining 0 and $1 + i$.

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1.7 Prove that $\oint_C \frac{\mathrm{d}z}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n=2,3,4,\ldots \end{cases}$

where C is a simple closed curve bounding a region having $z=a$ as interior point. What is the value of the integral if $n = 0, -1, -2, -3, \ldots$?

1.8 Evaluate $\oint_C \frac{\mathrm{d}z}{z-2}$ where C is

(a) the circle $|z|=1$,

(b) the circle $|z+3i|=5$.

1.9 Evaluate

(a) $\oint_C \frac{\sin(z/2)}{z-\pi}\mathrm{d}z$, (b) $\oint_C \frac{\exp(2z)}{z(z+1)}\mathrm{d}z$

where C is the circle $|z-1|=4$.

1.10 Evaluate $\oint_C \frac{3z^2-17z+5}{(z-1)^3}\mathrm{d}z$ where C is any simple closed curve enclosing $z=1$.

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1.11 Evaluate $\int\limits_0^\infty \frac{dx}{x^4+1}$.

1.12 Show that $\int\limits_0^\infty \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$.

1.13 Evaluate $\int\limits_0^{2\pi} \frac{d\,\theta}{2+\cos\theta}$.

1.14 Evaluate $\int\limits_0^{2\pi} \frac{d\,\theta}{5+3\sin\theta}$.

1.15 Show that $\int\limits_{-\infty}^{\infty} \frac{e^{j\omega t}dt}{t^2+\tau^2} = \frac{\pi}{\tau}e^{-\omega\tau}$.

1.16 Show that $\int\limits_0^\infty \frac{\cos mx\,dx}{x^2+1^2} = \frac{\pi}{2}e^{-m},\ m>0$.

1.17 Show that $\int\limits_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

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1.18 Evaluate $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$.

1.19 Show that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

1.20 Prove that $\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos(\pi a/2)}$,

where $|a| < 1$.

1.21 Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.