<u>Multiple Integrals</u>

1D integrals: see page 5 of Spiegel.

<u>Double integrals</u>

Sand in a box has height $h(x, y)$.
What is the total volume of sand?

Divide the sand box into boxes of volume approximately given by

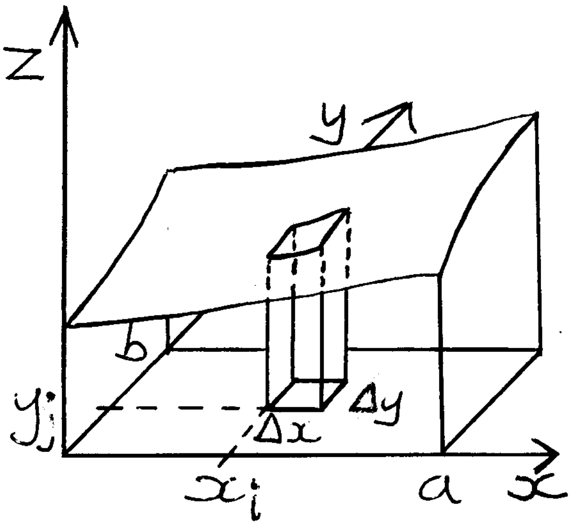
$$\Delta V = h(x_i, y_j)\Delta x \Delta y = h(x_i, y_j)\Delta A.$$

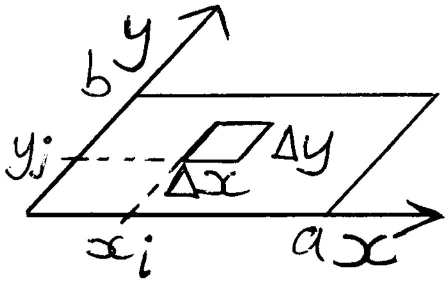
(x_i, y_j) are the coordinates of the corner of a particular box.

Then sum the volumes of the small boxes

$$S = \sum_{i,j} h(x_i, y_j)\Delta A = \sum_{i,j} h(x_i, y_j)\Delta x_i \Delta y_j .$$

Reduce ΔA for a better approximation.





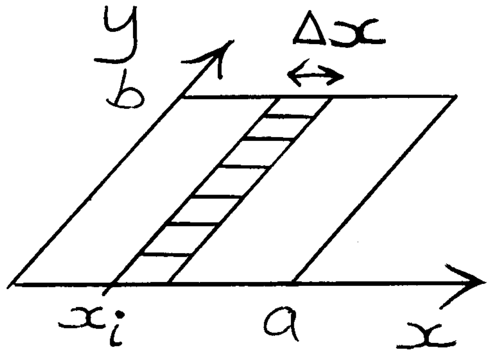
A sheet has mass $m(x,y)$ per unit area. What is the mass of the sheet?

Divide the sheet into squares of mass approximately equal to $\Delta M = m(x_i, y_j)\Delta A$.

Then sum over the different squares

$$S = \sum_{i,j} m(x_i, y_j)\Delta A = \sum_{i,j} m(x_i, y_j)\Delta x_i \Delta y_j \ .$$

How can we evaluate the sum?



One way is to sum all the elements for which $x = x_i$ and obtain the total for a strip of width Δx_i. Next add up the sums for the different strips,

$$S = \sum_i \left(\sum_j m(x_i, y_j)\Delta y_j \right)\Delta x_i \, .$$

As Δy_i is made infinitesimally small the summation over j becomes an integral

$$S = \sum_i \left(\int_0^b m(x_i, y)dy \right) \Delta x_i .$$

As Δx_i is made infinitesimally small the summation over i becomes an integral

$$S = \int_0^a \int_0^b m(x, y)dydx$$

We must perform a 1 dimensional integral with respect to y and then a 1 dimensional integral with respect to x. This is an *iterated integral*.

Notation: The limits written above the inner integral sign refer to the inner differential. i.e. in this case 0 and b are the limits for the integration with respect to y.

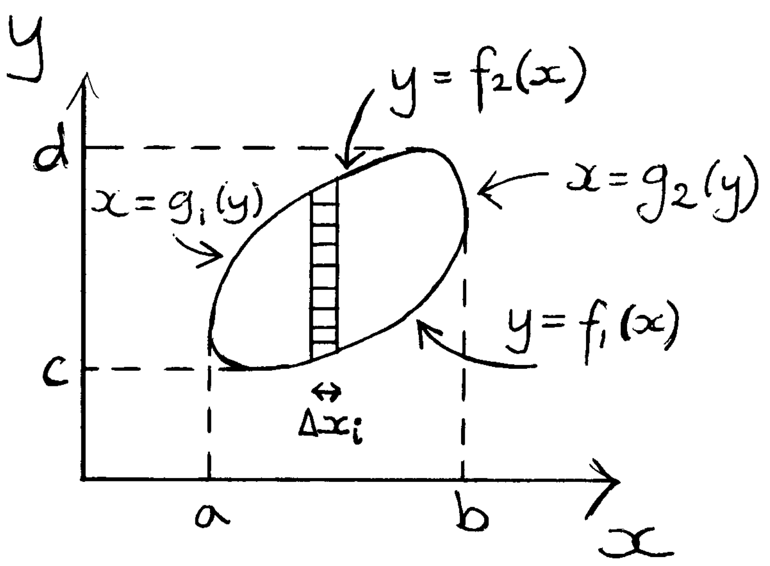
Example: evaluate the integrals

$$I = \int_0^a \int_0^b x\, e^{\beta y} dydx \quad \text{and} \quad I = \int_0^b \int_0^a x\, e^{\beta y} dxdy$$

Choosing the limits

Suppose that the sheet of material is not rectangular.

In general the limits for the integral with respect to y depend upon the value of x_i.



The integral is written as

$$I = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} m(x,y)\,dy\,dx.$$

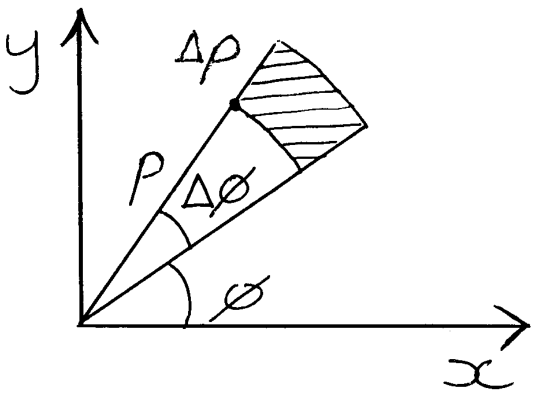
Changing the order of integration changes the limits - divide the region of integration into strips to remember how! Then

$$I = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} m(x,y)\,dx\,dy.$$

Changing coordinate system

A rectangular element of area with sides Δx_i and Δy_i is not the only choice.

What element of area might we use if we were working in 2D polar coordinates?



Subtracting the areas of sectors from two concentric circles of radius ρ and $\rho + \Delta\rho$.

$$\begin{aligned}\Delta A &= \frac{\Delta\phi}{2\pi}\left[\pi(\rho + \Delta\rho)^2 - \pi\rho^2\right] \\ &= \frac{\Delta\phi}{2}\left[\rho^2 + 2\rho\Delta\rho + (\Delta\rho)^2 - \rho^2\right] \\ &\approx \rho\Delta\rho\Delta\phi\end{aligned}$$

(as if the shaded area is a square of with sides $\Delta\rho$ and $\rho\Delta\phi$)

So, as $\Delta\rho$ and $\Delta\phi$ become infinitesimally small we write $dA = \rho\, d\rho\, d\phi$.

Example: Use both Cartesian and 2D polar coordinates to calculate the area of a circle of radius a.

Double integrals may be written as

$$\iint_{\Re} f \, dA$$

where $\Re$ is the region of integration.

The value of the integral is independent of the coordinate system used to evaluate it.

We choose the most convenient coordinate system - that which allows the integrand or limits to be written simply.

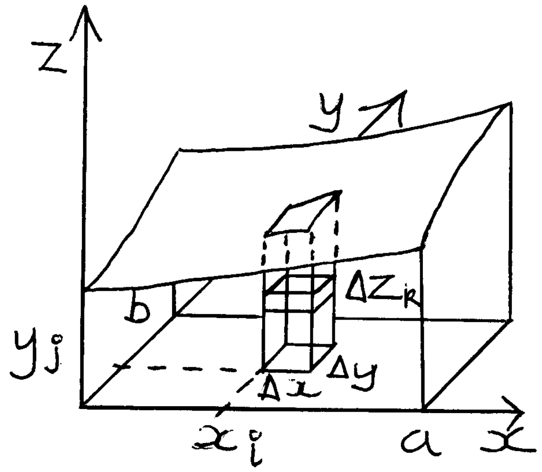
Example: A disc of radius a centred at the origin has mass per unit area

$$m = c\sqrt{a^2 - x^2 - y^2}\,.$$

Find the total mass of the disc.

<u>Triple Integrals</u>

A box of sand could be divided into boxes of volume $\Delta V = \Delta x\,\Delta y\,\Delta z$



The total volume is approximately the sum of the volumes of all the boxes.
Or if the density of the sand is $m(x, y, z)$ then the total mass is approximately

$$
\begin{aligned}
S &= \sum m(x, y, z)\Delta V \\
&= \sum_i \sum_j \sum_k m(x_i, y_j, z_k)\Delta z_k \Delta y_j \Delta x_i
\end{aligned}
$$

As $\Delta x_i, \Delta y_j, \Delta z_k$ become infinitesimally small the summations become integrals

$$M = \int_0^a \int_0^b \int_0^{h(x,y)} m(x, y, z)\,dz\,dy\,dx = \iiint_{\Re} m\,dV$$

The final expression is a *triple integral* in which the coordinate system is not specified but $\Re$ is the volume occupied by the sand.

Example: A sand box has a base

$$0 \le x \le a, \quad 0 \le y \le b, \quad z = 0.$$

The height of the sand in the box is

$$h(x, y) = xy$$

and its density is

$$m(x, y, z) = m_0 (c - z)$$

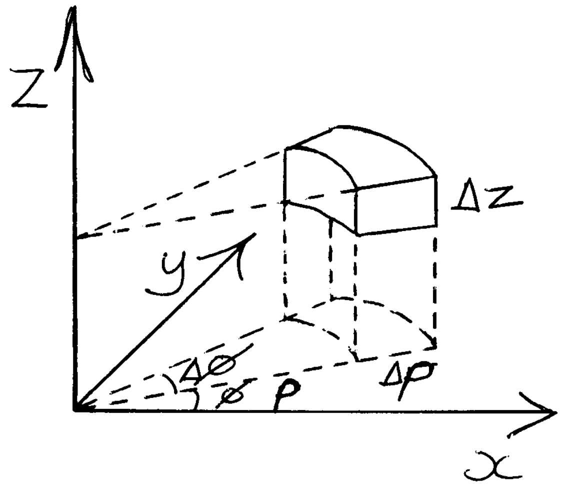
where c and m_0 are constants.

Find the total mass of sand in the box.

Changing coordinate system

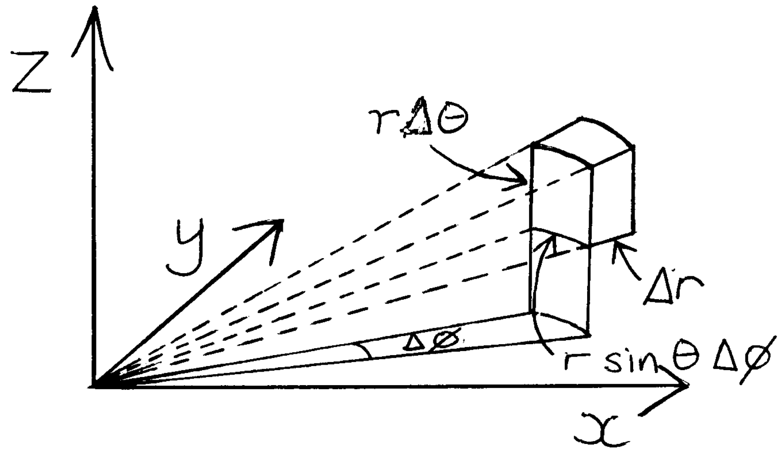
What element of volume do we use?

Cylindrical polar coordinates :



$$\Delta V \approx \rho\,\Delta\rho\,\Delta\phi\,\Delta z \quad \text{so} \quad dV = \rho\, d\rho\, d\phi\, dz.$$

Spherical polar coordinates:



$$\Delta V \approx r^2 \sin\theta\,\Delta r\,\Delta\phi\,\Delta\theta$$

and so $\quad dV = r^2 \sin\theta\, dr\, d\phi\, d\theta.$

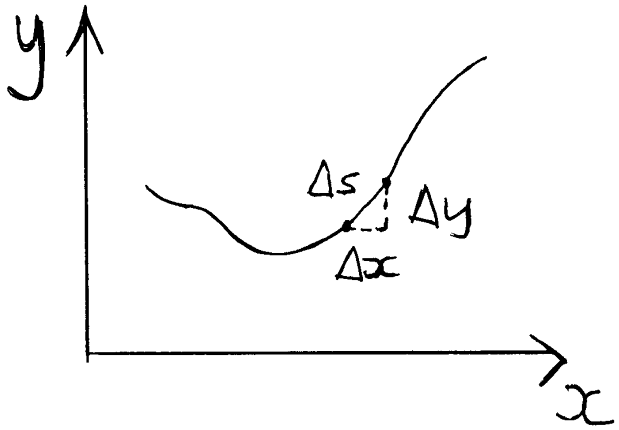
We use the coordinate system which is most convenient.

Example: Use both Cartesian and Sperical polar coordinates to calculate the volume of a sphere of radius a.

Line Integrals

For a piece of wire we have equations for:
(i) the curve describing its position.
(ii) its density at different points.

Could we then find its length and mass?



An infinitesimal segment of the curve has length ds and mass mds where m is the mass per unit length. From Pythagoras Theorem

$$ds = \left((dx)^2 + (dy)^2\right)^{\frac{1}{2}}.$$

The length ofwire between two points A and B on the curve is then given by

$$L = \int_C ds = \int_C \sqrt{(dx)^2 + (dy)^2}$$

where C is the arc joining A and B.

The mass of the wire between A and B is

$$M = \int_C m(x,y)\,ds$$

L and M are given by *line integrals* - 1D integrals along a curved path

Example: a piece of wire lies in the xy plane and has a shape described by the equation

$$y = a\,x^{3/2}, \text{ for } 0 \le x \le b.$$

The mass per unit length of the wire is given by

$$m = cx.$$

Find the length and mass of the wire.

The work done by a force **F** on a particle as it moves through a distance **dr** is given by $dW = \mathbf{F}\cdot\mathbf{dr}$. The total amount of work done in moving along a curve C is

$$
\begin{aligned}
W &= \int_C dW \\
&= \int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_C \left(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\right)\cdot\left(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right) \\
&= \int_C \left[F_1 dx + F_2 dy + F_3 dz\right]
\end{aligned}
$$

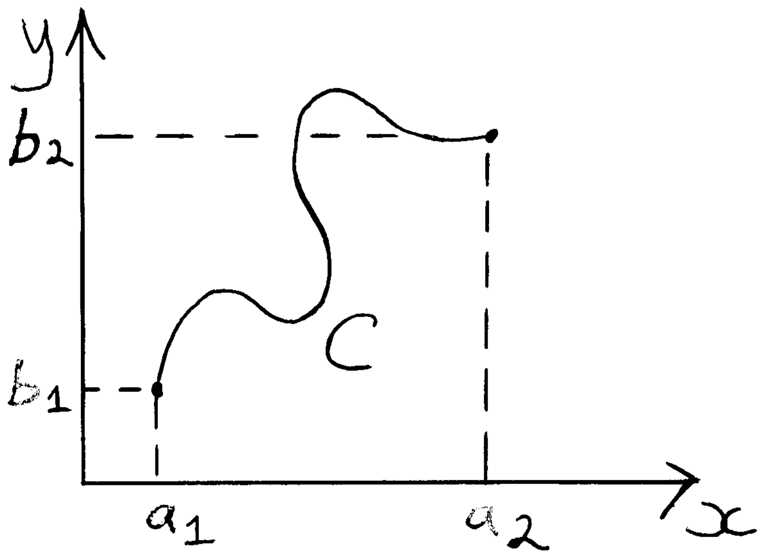
This is a *line integral* along a curve in 3 dimensional space.

From a purely mathematical point of view we may consider integrals of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

in which P, Q and R, have no physical meaning but these are of less interest to us here.

Parameterisation of line integrals
Line integrals may be evaluated by parameterisation (as used for the curved wire in an earlier example).



Consider the line integral

$$I = \int_C F(x, y)\, ds$$

Transforming ds, the integral becomes

$$I = \int_{a_1}^{a_2} F(x, y(x)) \left(1 + \left(\frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} dx,$$

so that here x is the parameter.
Or we could choose y to be the parameter:

$$I = \int_{b_1}^{b_2} F(x(y), y) \left(\left(\frac{dx}{dy} \right)^2 + 1 \right)^{\frac{1}{2}} dy.$$

The position on the path might be described by a separate parameter such as the time t. The integral then becomes

$$I = \int_{t_1}^{t_2} F(x(t), y(t)) \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right)^{\frac{1}{2}} dt$$

For integrals of the type

$$I = \int_C \left[P(x,y)dx + Q(x,y)dy \right]$$

similar parameterisations can be made:

$$I = \int_{a_1}^{a_2} \left[P(x, y(x)) + Q(x, y(x)) \frac{dy}{dx} \right] dx \, .$$

or

$$I = \int_{b_1}^{b_2} \left[P(x(y), y) \frac{dx}{dy} + Q(x(y), y) \right] dy \, .$$

or

$$I = \int_{t_1}^{t_2} \left[P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right] dt \, .$$

We will usually choose the parameterisation that leads to the simplest form for the integrand.

Example: Evaluate the integral $\int_C xy\,ds$ where C is the contour made up of straight line segments joining the points $(a,0)$, (a,a), and $(-a,0)$.

Example: Evaluate the integral $\int_C x^2 ds$ where C is a semicircular arc in the upper half plane, centred on the origin with radius a. The integral is evaluated from $(a,0)$ to $(-a,0)$.

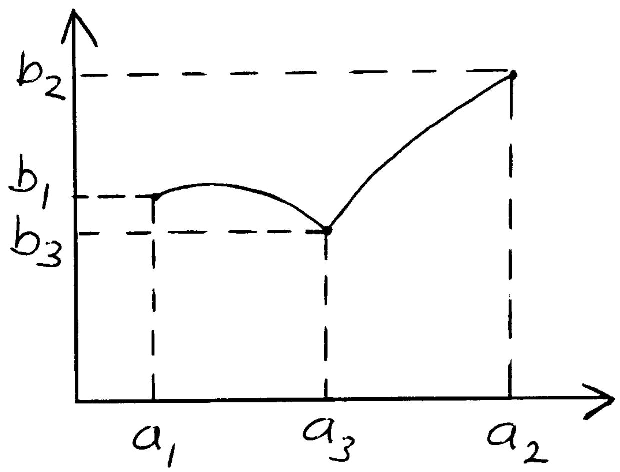
Properties (Spiegel, page 151)

1. $$\int_C [P(x,y)dx + Q(x,y)dy] = \int_C P(x,y)dx + \int_C Q(x,y)dy$$

2. Reversing the direction of integration along C changes the sign of the integral.

$$\int_{(a_1,b_1)}^{(a_2,b_2)} [Pdx + Qdy] = -\int_{(a_2,b_2)}^{(a_1,b_1)} [Pdx + Qdy]$$

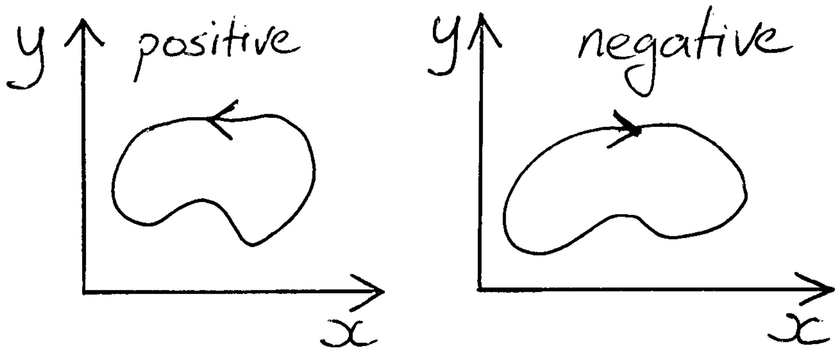
3. Line integrals may be split up



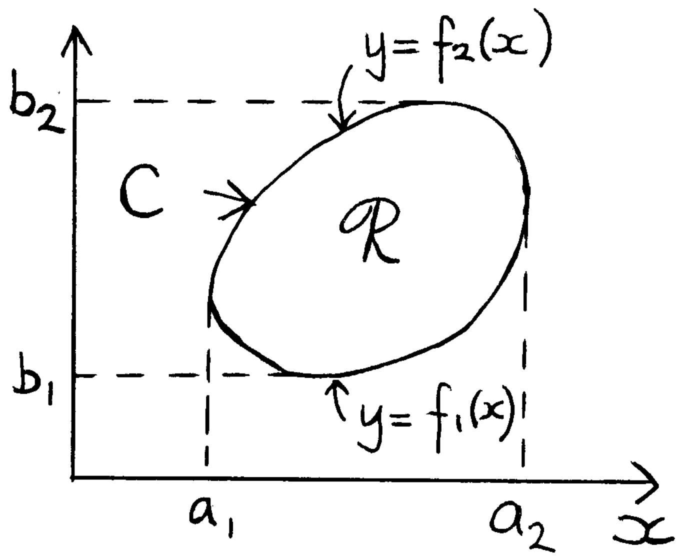
$$\int_{(a_1,b_1)}^{(a_2,b_2)} [Pdx + Qdy] = \int_{(a_1,b_1)}^{(a_3,b_3)} [Pdx + Qdy] + \int_{(a_3,b_3)}^{(a_2,b_2)} [Pdx + Qdy]$$

Different parameterisations may then be used for the two parts.

4. A line integral around a *closed* curve has a 'sense'.



Green's Theorem in the plane



The region $\Re$ is bounded by the closed curve C. Consider the 2D integral

$$I = \iint_{\Re} \frac{\partial P}{\partial y}\, dx\, dy$$

in which P is some function of x and y. The curve C can be broken into two parts $y = f_1(x)$ and $y = f_2(x)$ for $a_1 \le x \le a_2$.

Integrating first with respect to y the limits are easily written:

$$I = \int_{a_1}^{a_2} \int_{f_1(x)}^{f_2(x)} \frac{dP}{dy} dydx$$

$$= \int_{a_1}^{a_2} \left[P\left(x, f_2(x)\right) - P\left(x, f_1(x)\right)\right] dx \quad .$$

This can be split up into line integrals along the top and bottom parts of C.

$$I = \int_{a_1}^{a_2} P(x, f_2(x))dx - \int_{a_1}^{a_2} P(x, f_1(x))dx$$

$$= -\int_{a_2}^{a_1} P(x, f_2(x))dx - \int_{a_1}^{a_2} P(x, f_1(x))dx \quad .$$

$$= -\oint_C Pdx$$

The circle on the integral sign indicates that the curve is closed and the integral is understood to be evaluated in the positive sense. Our result is that

$$\iint_{\Re} \frac{\partial P}{\partial y} dx\, dy = -\oint_C P\, dx,$$

which is one form of Green' s theorem in the plane.

Alternatively, for Q, a second function of x and y, we could integrate w.r.t. x first and obtain

$$\iint_{\Re} \frac{\partial Q}{\partial x}\, dxdy = \int_{b_1}^{b_2} \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x} dxdy$$

$$= \int_{b_1}^{b_2} \left[Q\left(g_2(y), y\right) - Q\left(g_1(y), y\right) \right] dy$$

$$= \int_{b_1}^{b_2} Q\left(g_2(y), y\right) dy + \int_{b_2}^{b_1} Q\left(g_1(y), y\right) dy$$

$$= \oint_C Q dy$$

Notice the sign is different! Adding the two equations together we obtain:

$$\oint_C (P\, dx + Q\, dy) = \iint_{\Re} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

which is another statement of Green's Theorem in the plane.

Example: Calculate the area of an ellipse that has major axes of length $2a$ and $2b$

Line integrals that are independent of path

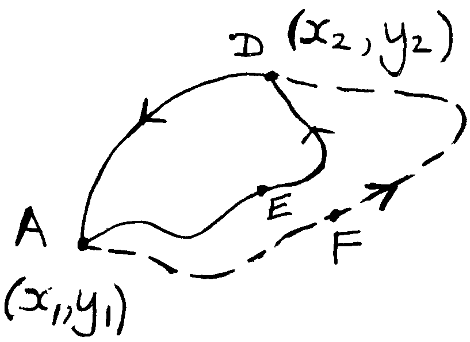
From Green's Theorem, if throughout $\Re$

(1) $$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

then for *any* closed curve C in $\Re$

(2) $$\oint_C (P\,dx + Q\,dy) = 0.$$

Also if (2) is true for *every* closed curve in $\Re$ then (1) is true throughout $\Re$. Then:



$$\oint_C = 0 = \int_{AED} + \int_{DA} = \int_{AFD} + \int_{DA}$$

and so $$\int_{AED} = \int_{AFD}.$$

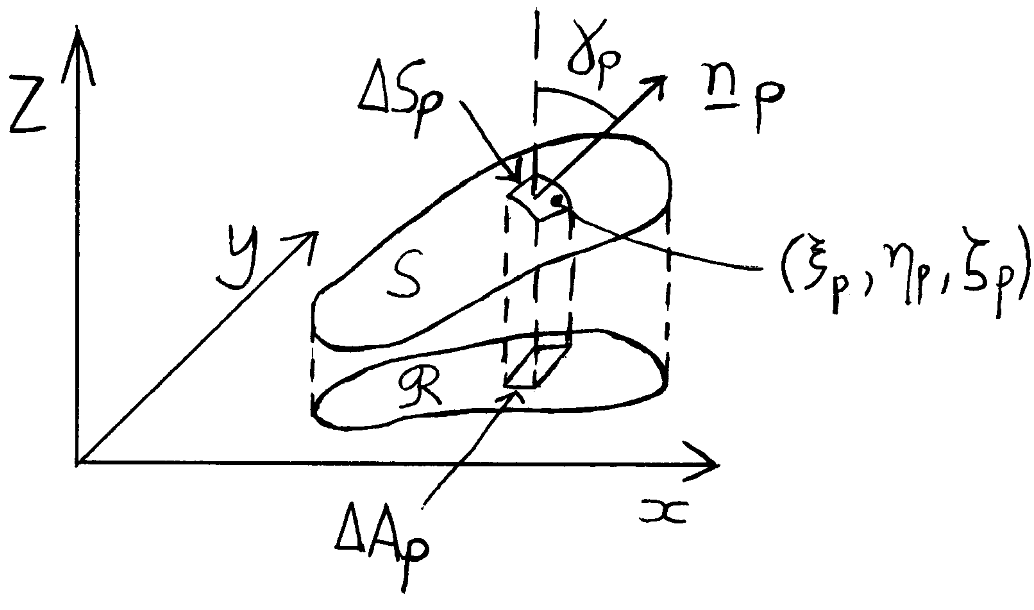
The integral from A to D is independent of the path and represents the change in a function $Z(x,y)$ where $dZ = P\,dx + Q\,dy$. Equation (1) is just the condition for this differential to be exact. e.g. P and Q could be the x and y components of a *conservative* force.

Surface Integrals

Line integrals along a curve are the generalisation of a 1D integral.

2D integrals can be generalised to integrals over curved surfaces.

Consider a surface S defined by the equation $z = h(x,y)$. If the mass per unit surface area is given by $m(x,y,z)$ then how do we calculate the total mass?



S can be divided up into elements of area ΔS_p. If (x_p, y_p, z_p) is a point on the element of area then the mass of the element is approximately

$$\Delta M_p = m\left(x_p, y_p, z_p\right)\Delta S_p .$$

So the total mass is

$$M \approx \sum_p \Delta M_p = \sum_p m\left(x_p, y_p, z_p\right)\Delta S_p$$

As ΔS_p is made infinitesimally small, the summation tends to the surface integral

$$M = \iint_S m(x,y,z)\,dS.$$

S can be projected onto a region $\Re$ in the xy plane. The vector $\mathbf{n}_p$ is the normal to the surface at a given point. The area of the projection of ΔS_p onto the xy plane is

$$\Delta A_p = \left|\cos\gamma_p\right|\Delta S_p$$

where γ_p is the angle between $\mathbf{n}_p$ and the z axis i.e. $\cos\gamma_p = \mathbf{n}_p \cdot \mathbf{k}$. Then

$$M = \iint_S m(x,y,z)\,dS = \iint_{\Re} m(x,y,z)\left|\sec\gamma\right| dA$$

in which $z = h(x,y)$.

To find $\mathbf{n_p}$ and hence $\left|\sec\gamma\right|$ we first find two vectors parallel to the surface. If we hold y constant and change x by Δx, then z will change by $\left(\frac{\partial z}{\partial x}\right)_y \Delta x$ and so the vector $\left(\Delta x, 0, \left(\frac{\partial z}{\partial x}\right)_y \Delta x\right)$ lies parallel to the surface. Two such unit vectors are

$$\mathbf{u_1} = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)_y^2}}\left(1, 0, \left(\frac{\partial z}{\partial x}\right)_y\right)$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial y}\right)_x^2}}\left(0, 1, \left(\frac{\partial z}{\partial y}\right)_x\right)$$

The unit vector perpendicular to these is

$$\mathbf{n_p} = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)_y^2+\left(\frac{\partial z}{\partial y}\right)_x^2}}\left(-\left(\frac{\partial z}{\partial x}\right)_y, -\left(\frac{\partial z}{\partial y}\right)_x, 1\right)$$

and so now we may calculate

$$\cos\gamma_p = \mathbf{n}_p \cdot \mathbf{k} = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)_y^2+\left(\frac{\partial z}{\partial y}\right)_x^2}}$$

The integral then becomes:

$$M = \iint_S m(x,y,z)\,dS$$

$$= \iint_{\Re} m(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

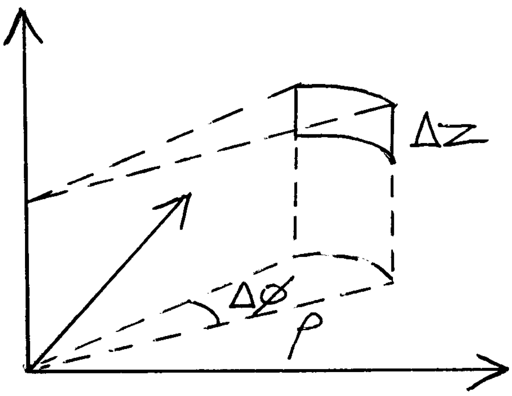
The surface integral has been converted into a double integral over x and y, which we know how to evaluate.

Example: Evaluate $I = \iint_S x^2\, dS$, where S is the hemisphere defined by

$$x^2+y^2+z^2=a^2,\ z \geq 0.$$

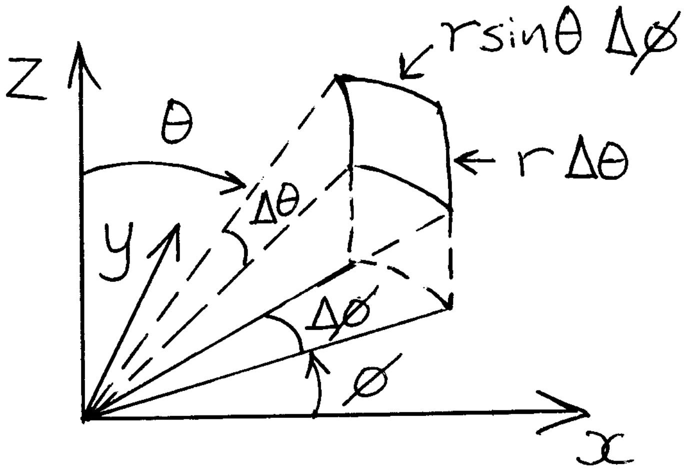
Sometimes it is easier to use a non-Cartesian coordinate system. But we need an expression for dS.

For a cylindrical surface .



$$dS = \rho \, d\phi \, dz$$

For a spherical surface



$$dS = r^2 \sin\theta \, d\theta \, d\phi$$

Example: we will recalculate the integral from the previous problem using spherical polar coordinates.

Example: Evaluate $I = \iint_S x^2 dS$ where S is now the cylindrical surface defined by $\rho = a,\ -b \le z \le +b$.

The Dirac Delta Function

If the density of material in a volume V is $m(x,y,z)$ then the total mass is

$$M = \iiint_V m\, dV .$$

But what is the density of a particle, such as an electron, which is a point mass?

Consider a 'top hat' that has unit area.

$$f(x) = \begin{cases} \dfrac{1}{2w}, & X - w < x < X + w \\ 0, & x < X + w,\, x > X + w \end{cases}$$

As $w \to 0$ the 'top hat' function tends to the *Dirac Delta Function*, which we write as $\delta(x - X)$.

The Dirac Delta Function is *defined* so as to have the following properties:

$\delta(x - X) = 0$ for $x \neq X$, and

$$\int_{-\infty}^{\infty} \delta(x - X)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - X) dx = f(X).$$

However there is no explicit expression for the delta function.

Returning to the case of the electron which has mass M_e, in a 1D problem, let the electron be located at $x = X$. If we write the density as

$$m(x) = M_e \delta(x - X)$$

then there is mass only at $x = X$ as required and the total mass is

$$M = \int_{-\infty}^{+\infty} m\,dx = M_e \int_{-\infty}^{+\infty} \delta(x - X)dx = M_e$$

In 3 D, if the electron is located at $\mathbf{r} = \mathbf{R} = (X, Y, Z)$ then we could write

$$\begin{aligned} m(\mathbf{r}) &= M_e \delta(\mathbf{r} - \mathbf{R}) \\ &= M_e \delta(x - X)\delta(y - Y)\delta(z - Z) \end{aligned}.$$

Again there is mass only at the point $\mathbf{r} = \mathbf{R}$ and the total mass is given by

$$\begin{aligned} M &= \iiint m\,dV \\ &= M_e \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x - X)\delta(y - Y)\delta(z - Z)dx\,dy\,dz \\ &= M_e \int_{-\infty}^{+\infty} \delta(x - X)dx \int_{-\infty}^{+\infty} \delta(y - Y)dy \int_{-\infty}^{+\infty} \delta(z - Z)dz \\ &= M_e \end{aligned}$$

Other Properties

1. $$\delta(-x) = \delta(x)$$

2. $$\delta[a(x-X)] = \frac{1}{|a|}\delta(x-X)$$

3. $$\frac{d}{dx}\delta(x) = -\frac{1}{x}\delta(x)$$

4. $$\int_{-\infty}^{+\infty}\delta'(x)f(x)dx = -f'(0)$$

The top hat is not the only way of approximating the delta function. The following functions may also be used in the limit that $n \to \infty$:

$$\delta(x) = \lim_{n\to\infty}\frac{n}{\sqrt{\pi}}\exp\left(-n^2x^2\right) \qquad \text{(Gaussian)}$$

$$\delta(x) = \lim_{n\to\infty}\frac{n}{\pi\left(1+n^2x^2\right)} \qquad \text{(Lorentzian)}$$

$$\delta(x) = \lim_{n\to\infty}\frac{\sin nx}{\pi x}$$

Example: Evaluate $\int_V r^2 \delta(\mathbf{r} - \mathbf{r_0}) dV$

where $\mathbf{r_0} = (2,0,1)$.

Example: Show that

$$\int_{-\infty}^{+\infty} \delta'(x) f(x) dx = -f'(0)$$