Fourier Analysis Problem Set 2

1. The Fourier transform and the inverse Fourier transform are defined as

$$\mathcal{F}\{f\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\alpha u}\mathrm{d}u \quad \text{and} \quad \mathcal{F}^{-1}\{F\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha u}\mathrm{d}\alpha$$

respectively. Find the Fourier transform of

$$f(x)=\begin{cases} +1, & 0\le x<1 \\ -1, & -1<x<0 \\ 0, & |x|\ge 1 \end{cases}.$$ [5]

2. The Fourier transform of

$$f(x)=\begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\ge 1 \end{cases}$$

is given by

$$F(k)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin k-k\cos k}{k^3}\right).$$

Write down an expression for the inverse Fourier transform of $F(k)$. Explain why the imaginary part of the integral vanishes, and by considering the case $x=\frac{1}{2}$, show that

$$\int_0^{\infty}\left(\frac{\sin k-k\cos k}{k^3}\right)\cos\left(\frac{k}{2}\right)dk=\frac{3\pi}{16}.$$ [5]

3. Let $F(k)$ be the Fourier transform of $f(x)$ and $G(k)$ be the Fourier transform of $g(x)=f(x+A)$. Show that

$$G(k)=e^{-ikA}F(k).$$

4. If the function $f(x)\to 0$ as $x\to\pm\infty$, show that

$$\mathcal{F}\left\{\frac{d^n f}{dx^n}\right\}=(-i\alpha)^n\mathcal{F}\{f\}$$

where $\mathcal{F}\{f\}$ is the Fourier transform of f as defined in question 1. [4]

5. (a) Show that the Fourier transform of the function

$$f(x) = \begin{cases} 1, & -w < x < w \\ 0, & x < -w,\ x > w \end{cases}$$

is

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \mathrm{w} \operatorname{sinc}(\alpha \mathrm{w}).$$

(b) Show that the convolution $f * f$ is given by

$$f * f = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(2w - |x|\right), & -2w < x < 2w \\ 0, & x < -2w, x > 2w \end{cases}.$$

(c) Use the convolution theorem to find the Fourier transform $\mathcal{F}(f * f)$ and make a sketch showing where $\mathcal{F}(f * f)$ touches or crosses the horizontal and vertical axes. [13]

6. Use the convolution theorem to show that $f * g = g * f$. [5]