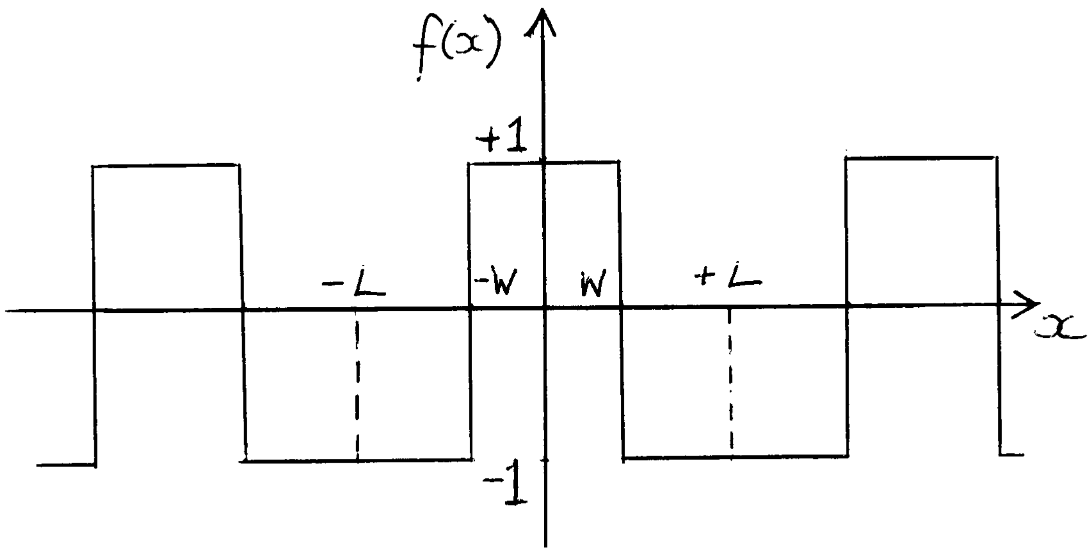
Fourier Transforms

So far we have only found Fourier series expansions for *periodic* functions. Suppose we have a function that is not periodic and wish to decompose it into a superposition of sine waves. How do we do it?

We will try to gain some insight into this problem by considering an example.

Let's reconsider a square wave that has positive and negative regions of different widths.



We will find the Fourier expansion for this function and then allow L to become large while w is held constant. In the limit that $L \to \infty$, $f(x)$ will consist of a single "top hat", and will no longer be a periodic function.

The Fourier expansion in the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

From the graph we see that $f(x)$ is an even function. So there can be no sine waves in the expansion, and,

$$b_n = 0.$$

Next we calculate

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \text{ for } n = 0,1,2\ldots,$$

which gives

$$a_0 = \frac{4w}{L} - 2;\ a_n = \frac{4}{n\pi}\sin\left(\frac{n\pi w}{L}\right),\ n = 1,2\ldots$$

Let's consider the case that $L = 2w$ then

$$a_0 = 0$$

$$a_{2m} = 0,$$

$$a_{2m-1} = \frac{4(-1)^{m-1}}{(2m-1)\pi},\ m = 1,2\ldots$$

i.e. $a_n = 0, \frac{4}{\pi}, 0, -\frac{4}{3\pi}, 0, \frac{4}{5\pi}, 0, -\frac{4}{7\pi}, \ldots \quad n = 0,1,\ldots$

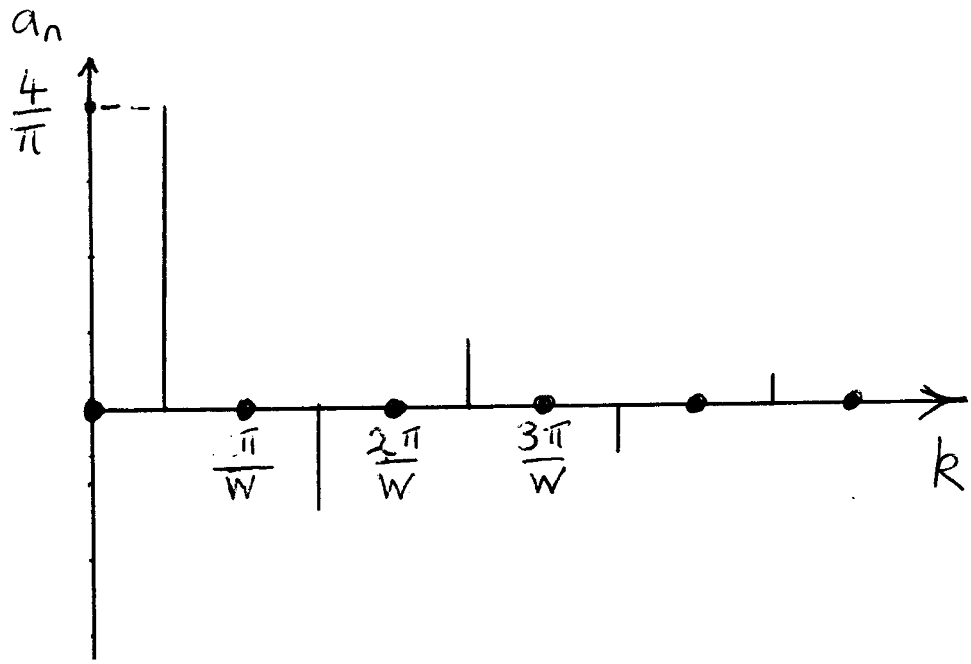
Let's introduce a new variable

$$k = \frac{n\pi}{L} = \frac{n\pi}{2w}.$$

Then we have

$$f(x) = \sum_{n=1}^{\infty} a_n \cos k_n x.$$

Now lets plot a_n as a function of k_n.

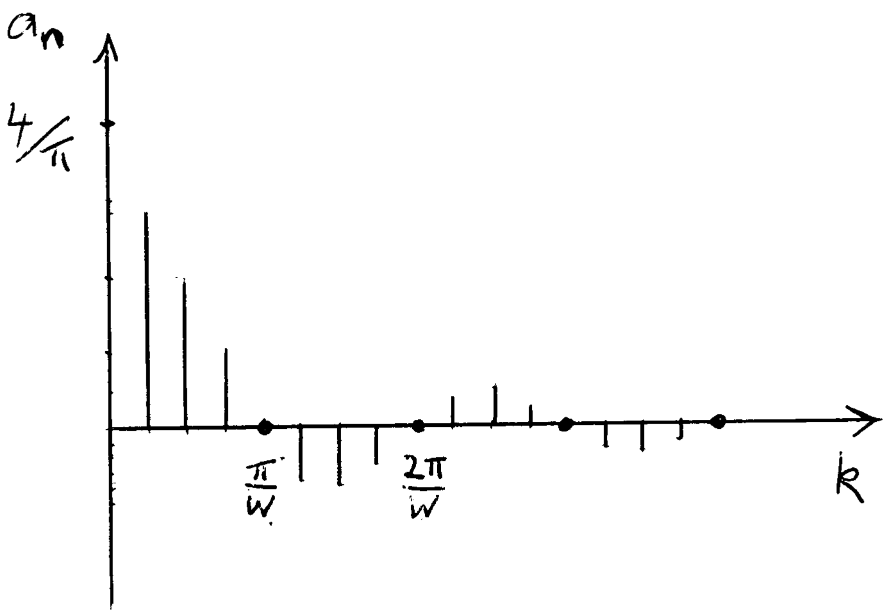


Next let us instead make $L = 4w$. Then

$$a_0 = -1,$$

$$a_n = \frac{4}{n\pi}\sin\left(\frac{n\pi}{4}\right),\ n = 1, 2\ldots$$

Plotting these coefficients against $k = \frac{n\pi}{L} = \frac{n\pi}{4w}$,

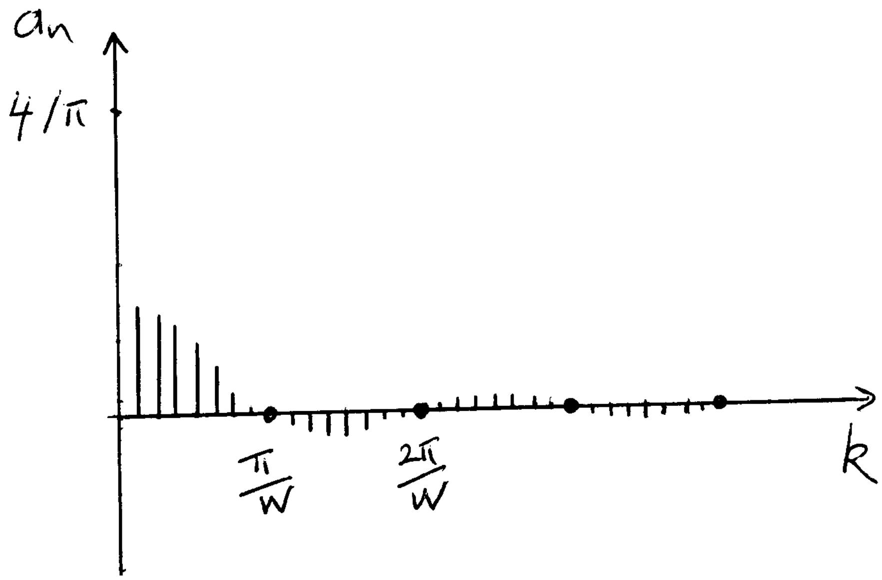


And now if we set $L = 8w$ then

$$a_0 = -\frac{3}{2},$$

$$a_n = \frac{4}{n\pi}\sin\left(\frac{n\pi}{8}\right)$$

Ploting the Fourier coefficients against $k = \frac{n\pi}{L} = \frac{n\pi}{8w}$ we obtain



If we were to continue to increase the ratio of L to w then we would find that:

- The crossing point of the "envelope" curve does not move on the k axis
- The magnitude of a_n decreases
- The density of points on the k axis continues to increase and tends towards a continuum of values

As we let L tend to ∞ we are making a Fourier expansion of a single "top hat" which will be valid for *all* values of x.

We can replace the summation by an integral

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \to \frac{a_0}{2} + \int_0^{\infty} A(k)\cos kx\,dk.$$

where

$$\begin{aligned} A(k) &= \frac{La(n)}{\pi} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos kx\,dx \end{aligned}$$

and $\frac{a_0}{2} = -1$ is the mean value of the function.

We have made a *Fourier integral expansion* of the function $f(x)$ and the function $A(k)$ is its*Fourier Transform* .

Fourier's Integral Theorem

In our example there were no sine terms because our original function was an even function. For a more general function let's try an expansion

$$f(x) = \int_0^{\infty} \{A(k)\cos kx + B(k)\sin kx\} dk$$

We can do this if

1. $f(x)$ satisfies the Dirichlet conditions for all values of x.
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges to a finite value. *This implies that the average value of the function must be zero!*

As for Fourier series, at a point of discontinuity the value of the Fourier expansion is the average of the values on either side of the discontinuity.

Can we find expressions for the *Fourier coefficients* $A(k)$ and $B(k)$?

We use a similar approach as for Fourier series: multiply both sides by $\cos k'x$ and integrate w.r.t. x between $\pm\infty$.

$$\int_{-\infty}^{+\infty} f(x)\cos k'x\,dx$$

$$= \int_{-\infty}^{+\infty}\int_{0}^{\infty}\{A(k)\cos kx + B(k)\sin kx\}dk\cos k'x\,dx$$

$$A(k)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos kx\,dx$$

and we made use of the result

$$\int_{-\infty}^{\infty}\cos\left((k-k')x\right)dx = 2\pi\delta(k-k')$$

If instead we were to multiply both sides by $\sin k'x$ and integrate then we could obtain

$$B(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin kx \, dx$$

In summary, the Fourier integral expansion of the function $f(x)$ is given by

$$f(x) = \int_0^{\infty} \{A(k) \cos kx + B(k) \sin kx\} dk$$

where

$$A(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos kx \, dx$$
$$B(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin kx \, dx$$

These latter expressions are in fact Fourier transforms of the function $f(x)$.

This form for the Fourier integral expansion illustrates the similarity to the Fourier series expansion. Notice how the Dirac delta function appeared in analogy to the Kronecker delta.

However this is not the form that is most commonly used. Rather we need to rewrite it in terms of exponential functions.

Exponential form of Fourier’s Integral Theorem

In the equations that we have just derived we may replace the sine and cosine functions with exponentials using the Euler formulae.

We then obtain

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk$$

where

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx$$

Actually we are free to share the factor of $\frac{1}{2\pi}$ between $f(x)$ and $F(k)$ as we like. A symmetric choice is probably the most common and from now on we will us the following definitions:

$F(k)$ as *The Fourier Transform* of $f(x)$ where

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk$$

and $f(x)$ is the *inverse Fourier Transform* of $F(k)$, where

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx$$

Example

Find the Fourier transform of

$$f(x)=\begin{cases}\dfrac{1}{2w}, & X-w<x<X+w\\ 0, & x<X-w,\ x>X+w\end{cases}.$$

Example

Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}.$$

The Dirac Delta Function

We found that the Fourier transform of a 'top hat' with area equal to unity is

$$F(k) = \frac{1}{\sqrt{2\pi}} \mathrm{sinc}(k\mathrm{w}) e^{ikX}.$$

If we squeeze the 'top hat', then as $w \to 0$, $f(x)$ tends to the Dirac delta function $\delta(x - X)$.

If we let $w \to 0$ in the expression for $F(k)$, we must obtain the Fourier Transform of $\delta(x - X)$.

So, as $w \to 0$, $\mathrm{sinc}(kw) \to 1$, for all values of k, and so

$$F(k) \to \frac{1}{\sqrt{2\pi}} e^{ikX}$$

In summary

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - X) e^{ikx}\, dx = \frac{1}{\sqrt{2\pi}} e^{ikX}.$$

The Fourier transform of the Dirac delta function is an exponential.

Consider next the inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk$$

which gives the useful result

$$\delta(x-X)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(X-x)}dk$$

We can show also that the Dirac delta function is the Fourier transform of an exponential function.

If $f(x)=\frac{1}{\sqrt{2\pi}}e^{iKx}$ then

$$\begin{aligned}F(k)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{iKx}e^{ikx}\,dx\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k+K)x}\,dx\\&=\delta(K+k)\end{aligned}$$

This should not surprise us: the exponential function represents a single plane wave.

The delta function allows us to pick out this single plane wave in the Fourier expansion as follows:

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(k)e^{-ikx}dk\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\delta(K+k)e^{-ikx}dk.\\&=\frac{1}{\sqrt{2\pi}}e^{iKx}\end{aligned}$$

Example

Consider the Gaussian function

$$f(x) = \exp\left(-\frac{x^2}{a^2}\right),$$

that has a half width at half maximum of

$$\Delta x = a\sqrt{\ln 2}.$$

The Fourier transform is given by

$$F(k) = \sqrt{\frac{\pi}{2}}\, a \exp\left(-\frac{a^2 k^2}{4}\right)$$

for which

$$\Delta k = \frac{2}{a}\sqrt{\ln 2}\,.$$

Hence we find that

$$\Delta x\ \Delta k = 2 \ln 2\,,$$

which is independent of the constant a that determines the width of both $f(x)$ and $F(k)$.

It is a general feature of Fourier transforms that if we make the function $f(x)$ wider, then $F(k)$ becomes narrower.

The Heisenberg Uncertainty Principle provides a physical example of this phenomenon.

In fact the Gaussian is rather special: it is the function that minimises the product $\Delta x \Delta k$.

The Convolution Integral

The *convolution* of two functions $f(x)$ and $g(x)$ is defined as

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)g(x-u)du$$

Example: Consider the functions

$$f(x)=\begin{cases}1, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

and $$g(x)=\begin{cases}1-x, & 0\le x\le 1\\ 0, & x<0,\, x>1\end{cases}$$

The convolution is given by

$$f*g=\begin{cases}0, & x>2\\ \dfrac{1}{2\sqrt{2\pi}}(x-2)^2, & 1<x\le 2\\ \dfrac{1}{2\sqrt{2\pi}}, & 0<x\le 1\\ \dfrac{1}{2\sqrt{2\pi}}\left(1-x^2\right) & -1<x\le 0\\ 0, & x\le -1\end{cases}$$

Does $f*g=g*f$?

The Convolution Theorem

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transform of $f(x)$ and the Fourier transform of $g(x)$.

$$\mathcal{F}\{f*g\}=\mathcal{F}\{f\}\mathcal{F}\{g\}$$

where $\mathcal{F}\{f\}$ denotes the Fourier transform of $f(x)$.