FOURIER ANALYSIS

(a) Fourier Series

(b) Fourier Transforms

Useful books:

1. ***"Advanced Mathematics for Engineers and Scientists", Schaum's Outline Series, M. R. Spiegel* -** The course text. We follow their notation but do not cover all material. Worked examples are useful

2. ***"Fourier Series", Library of Mathematics, RKP, I. N. Sneddon*** - More formal

3. ***"Mathematical Methods for Physicists", Arfken*** - good but more advanced

4. ***"Optics", Hecht and Zajac*** - Chapter 7, The Superposition of Waves

5. ***"University Physics", Young*** - for background on Waves and physical applications

What is Fourier Analysis and why do we need it?

In Physics we frequently (have found)/(will find) that the variation of a quantity in space and time is described by a sine wave. We call this a *harmonic* variation.

Examples:

- a simple harmonic oscillator such as a mass on a string - in fact most systems exhibit simple harmonic motion for small displacements!
- waves: light, sound, water waves...
- the wave function of a free particle in quantum mechanics

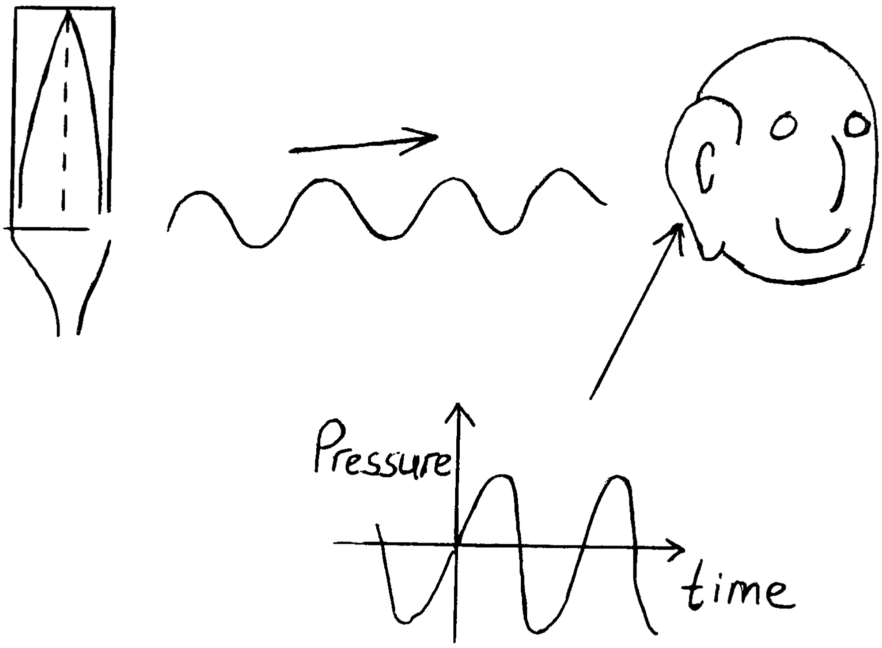
Also we have seen systems where the principle of *superposition* applies: when water waves meet, the height of the water is given by adding the amplitudes of the two waves together.

Superposition is obeyed in *linear* systems i.e. most of the systems that we will study in physics!

Fourier analysis is about using superposition to write functions in terms of the sine wave components that we are familiar with.

A Musical Example.

An organ pipe produces a pressure variation that varies as $\sin \omega t$. The frequency is
$f = \dfrac{\omega}{2\pi} = 262\text{Hz}$ - middle C on the piano

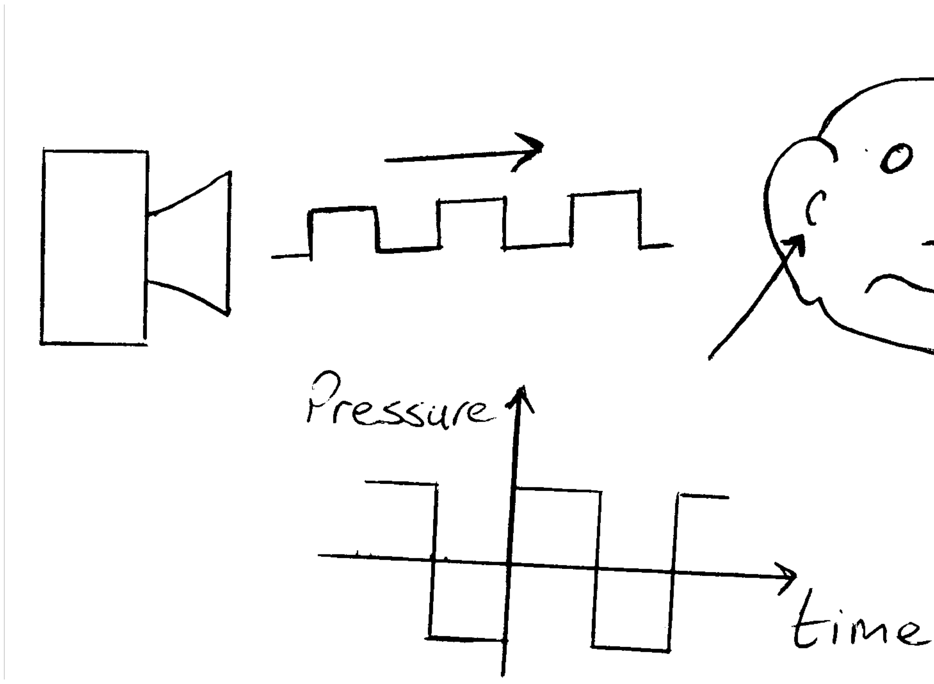


Sound travels to the ear of a listener who hears middle C.

One octave corresponds to a doubling of the frequency. The frequencies of C, E and G are in the ratio 3:4:5.

If 3 organ pipes play the chord C, E, G, the pressure variation is a *superposition* of three waves with different frequencies. Musical listeners can hear all three notes.

Suppose a physicist builds a device that produces a train of pressure pulses with frequency 262 Hz



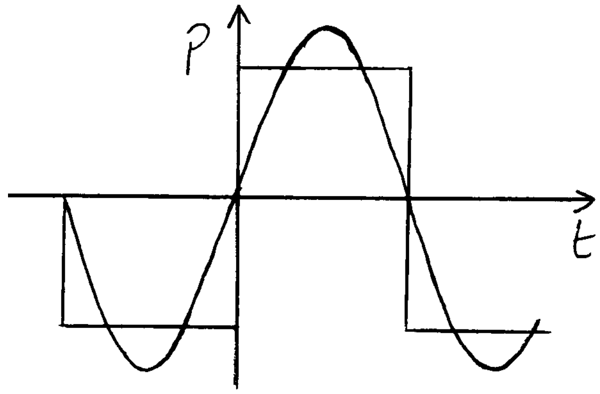
What does the listener hear now?

We know the ear detects sinusoidal (*harmonic*) variations of different frequency and sound waves obey the superposition principle.

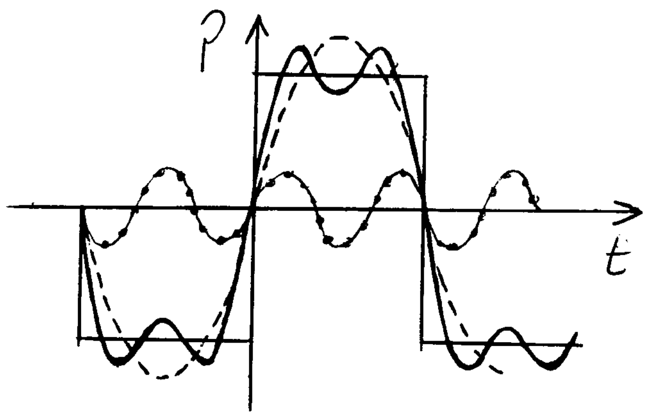
Can we think of the square wave as being made up of sine waves? What frequencies would these sine waves have?

Our first guess would be that there must be a sine wave present that has the same frequency as the square wave i.e. $\omega = \dfrac{2\pi}{262}$.

If the square wave has maximum and minimum values of +1 and -1, lets plot it against $\frac{4}{\pi}\sin\omega t$

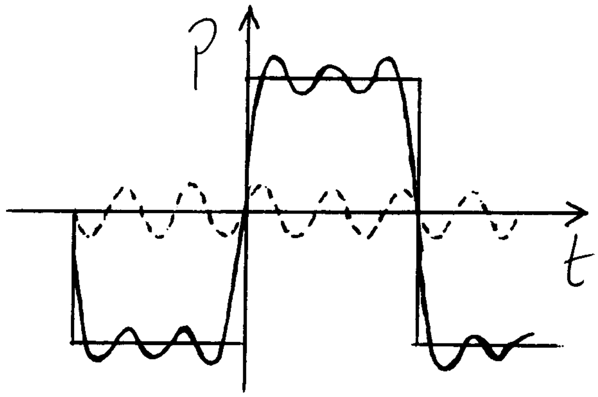


This is a reasonable first approximation but lets now add $\frac{4}{3\pi}\sin 3\omega t$



This looks to be a better approximation and maybe we can do better still.

Next we try adding $\frac{4}{5\pi}\sin 5\omega t$



If we were to add more sine waves we could exactly replicate the square wave!

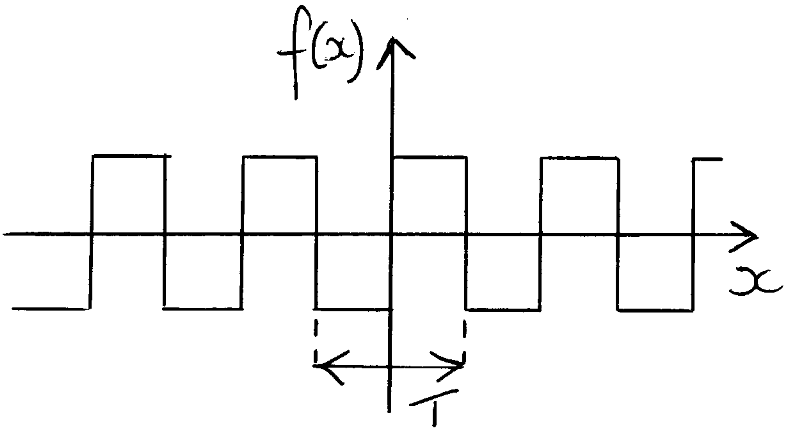
The listener hears a note of “pitch” middle C. This is the frequency of the lowest frequency mode. He also hears a number of “overtones”, i.e. the other sine wave components, which determine the quality or “timbre” of the sound.

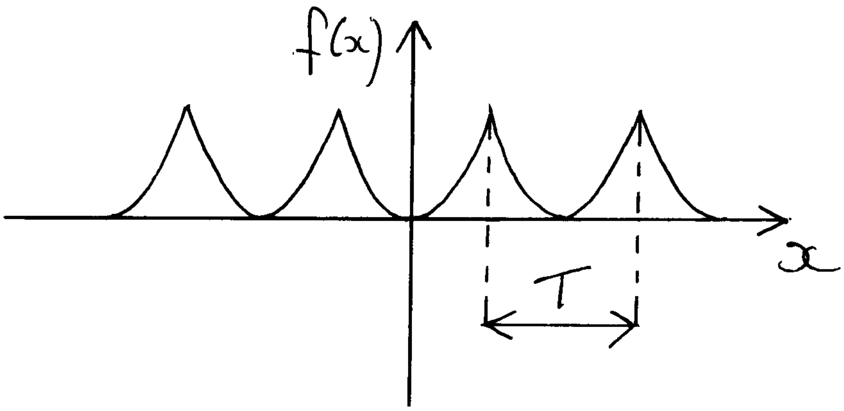
How do we know how to choose the frequencies and amplitudes of the sine waves that make up the square wave?

To answer this we need Fourier Analysis.

Periodic Functions

Examples:





The function $f(x)$ is periodic. It repeats itself after a distance T:

$$f(x+T)=f(x) \text{ for any } x.$$

T is the *period* of the function $f(x)$. Its frequency is $1/T$.

The Fourier Expansion

Let's consider a function $f(x)$ in the interval $(-L, L)$ that has period $2L$.

Let us try to build $f(x)$ out of sine and cosine functions. We try a form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Note that:

(i) In general we need both sine and cosines - the musical example was a special case

(ii) We do not yet know the values of the constants a_0, a_n, and b_n.

(iii) The first term $\frac{a_0}{2}$ is independent of x - it provides a constant vertical offset.

(iv) The arguments of the sine and cosine have been carefully chosen so that $f(x+2L) = f(x)$.

A *Fourier expansion* of $f(x)$ can be found as long as the *Dirichlet conditions* are satisfied.

Dirichlet conditions

(1) $f(x)$ is well defined and single-valued.

(2) $f(x)$ is periodic.

(3) $f(x)$ and $f'(x)$ are continuous except for finite discontinuities.

The Dirichlet conditions *will* be satisfied for all functions that we will consider. At a discontinuity the Fourier expansion gives the average of the value of the function on either side of the discontinuity

Now let us obtain the values of a_0, a_n, and b_n.

Take

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

and integrate both sides through $(-L, L)$. Then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$\frac{a_0}{2}$ is the mean value of $f(x)$ through $(-L, L)$.

Next multiply both sides by $\cos\frac{n\pi x}{L}$ and integrate through $(-L, L)$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \text{ for } n = 1,2\ldots$$

Multiply both sides instead by $\sin\dfrac{n\pi x}{L}$ and integrate through $(-L, L)$ to obtain

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \text{ for } n = 0,1,2\ldots$$

In summary:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

for $n = 0,1,2\ldots$

And we have made use of the following results:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}dx = \int_{-L}^{L} \sin\frac{n\pi x}{L}dx = 0$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = L\delta_{m,n}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = L\delta_{m,n},\ m,\ n = 1,2,3...$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

where the *Kronecker delta* has the value

$\delta_{m,n} = 1$ if $m = n$, and $\delta_{m,n} = 0$ if $m \neq n$.

Similarity to Vectors

In 3 dimensions we find it convenient to use a set of 3 *basis* vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ from which we can construct any other vector.

These are mutually *orthogonal* since

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{i,j}.$$

We can think of the values of $f(x)$ for points on the x axis in the interval $(-L, L)$ as the elements of an infinite vector.

The set of *basis* functions

$$\{\varphi_i\} = \left\{\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}, \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right\},$$

where $n = 1, 2\ldots$ are *orthonormal* since

$$\int_{-L}^{L} \varphi_i \varphi_j dx = \delta_{i,j}.$$

This set of functions can be used to construct any function that satisfies the Dirichlet conditions.

Sine and cosine waves are not the only possible basis functions, but they are probably the most useful functions for use in physics.

Example

Calculate the Fourier coefficients of

$$f(x) = \begin{cases} 1, & 0 < x < L \\ -1, & -L < x < 0 \end{cases}.$$

and hence show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots..$$

Notice that in the example since $a_n = 0$ there are no cosine terms in the Fourier expansion.

In this instance $f(x) = -f(-x)$ and so $f(x)$ is said to be an *odd* function.

If instead $f(x) = +f(-x)$ then $f(x)$ is said to be an *even* function.

The Fourier expansion of an odd function is a sum of sine functions which are themselves odd functions.

The Fourier expansion of an even function will be made up from cosine functions which are themselves even

This is a useful check when calculating Fourier expansions!

Example

Expand the function $f(x) = x^2, -L < x < L$ in a full range Fourier series. Hence show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots..$$

Example

Find the Fourier expansion of the function

$$f(x) = \begin{cases} \cos x & |x| < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} \le |x| \le \pi \end{cases}$$

in the range $-\pi \le x \le \pi$.

Exponential Notation

By writing the cosine and sine terms in terms of exponentials the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

can be rewritten as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx$$

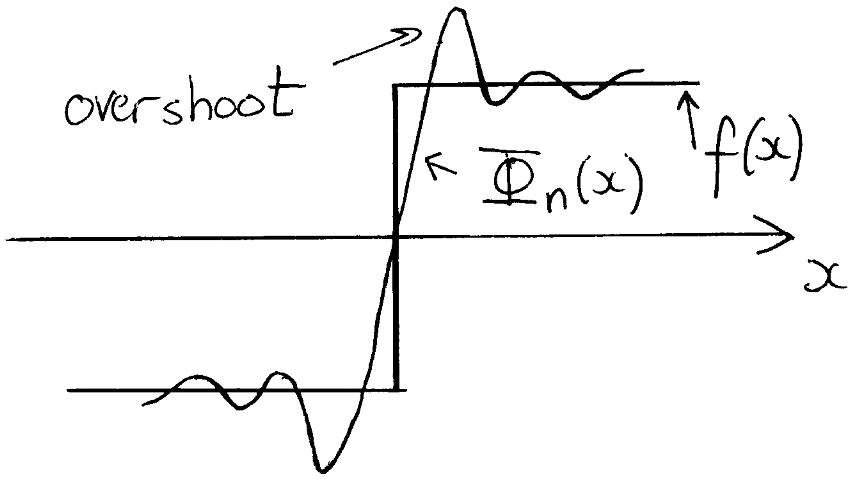
Truncating the Fourier Series

We may approximate $f(x)$ by taking only a finite number of terms in the Fourier series.

$$\Phi_m = \frac{a_0}{2} + \sum_{n=1}^{m}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

The Fourier series converges more slowly in the vicinity of a discontinuity in $f(x)$.

Therefore the truncated Fourier series Φ_m overshoots near a discontinuity. This is known as *Gibb's phenomenon.*



Retaining more terms does not remove the overshoot - rather the overshoot moves closer to the discontinuity in $f(x)$. The reason is that higher harmonics are required to accurately reproduce the shape of the discontinuity