Exercises 4

1. A rectangular sheet lies in the region $0 \le x \le 10$, $0 \le y \le 5$. If the mass per unit area of the sheet is given by
$$m(x,y) = xy$$
then find the total mass of the sheet.

2. A triangular sheet is bordered by the lines $x = 10$, $y = 0$, and $y = \frac{x}{2}$. If the mass per unit area of the sheet is given by
$$m(x,y) = xy$$
then find the total mass of the sheet.

3. (a) Using Cartesian coordinates, calculate the integral
$$I = \iint xy\,dx\,dy$$
over the quarter circle in the first quadrant that has radius a.

(b) Now use 2D polar coordinates to evaluate the integral.

Optional

4. The integral
$$I(a) = \int_{-\infty}^{+\infty} \exp\left(-ax^2\right)dx$$
occurs frequently in physics but is rather tricky to evaluate.

Begin by considering
$$[I(a)]^2 = \left(\int_{-\infty}^{+\infty} \exp\left(-ax^2\right)dx\right)\left(\int_{-\infty}^{+\infty} \exp\left(-ay^2\right)dy\right)$$
and then change to polar coordinates.

Exercises 5

1. A cube has corners at $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$. The density of the cube at any point is equal to the square of its distance from the origin. Find the mass of the cube.

2. Evaluate the integral $I = \iiint_{\Re} z\, dV$ where $\Re$ is a hemisphere of radius a which is bounded by and lies above the plane $z = 0$.

3. A piece of wire is bent into the shape of a parabola $y = ax^2$. If the ends of the wire are at $x = 0$ and $x = b$, and the mass per unit length of the wire is given by $m = cx$, then find the total mass of the wire.

Exercises 6

1. Evaluate the integral $\int_C \left(3x\,dx + xy^2\,dy\right)$ along the straight line joining the points $(0,0)$ and $(1,1)$.

2. A semicircular arc lies in the upper half of the xy plane and has end points at $(-a,0)$ and $(a,0)$. If the mass per unit length at a point (x,y) is given by

$$m = cy,$$

then find the mass of the wire.

3. Evaluate the line integral $\oint_C (x\,dy - y\,dx)$, where C is a square with vertices at $(a,a),(-a,a),(a,-a),(-a,-a)$. Also use Green's theorem to evaluate the integral and verify that its value is the same as before.

Exercises 7

1. Consider the surface S defined by $z = x^2$, $0 \le x \le 2$, $0 \le y \le 2$. Make a sketch of the surface. By projecting onto the xy plane, evaluate the integral $\iint_S xy^2 \, dS$.

2. Evaluate $\iint_S z \, dS$ where S is the hemisphere defined by
$x^2 + y^2 + z^2 = a^2, \quad z \ge 0$.

3. Evaluate $\iint_S x \, dS$ where S is the half cylinder defined by
$x^2 + y^2 = a^2, \quad x \ge 0, \quad 0 \le z \le 1$

Exercises 8

1. Evaluate the following integrals

(a) $\int_{-10}^{10} 5x^3 \delta(x-4)dx$ (b) $\int_{-10}^{0} 5x^3 \delta(x-4)dx$

(c) $\int_{-100}^{100} 5x^3 \delta(x+4)dx$ (d) $\int_{-100}^{100} 5x^3 \delta(2x-4)dx$

2. Make sketches of the 4 functions given in the lecture notes that can be used to approximate the delta function and label their height and width at half maximum.

3. By considering $\int_{-\infty}^{\infty} f(x)\delta(a(x-X))dx$, confirm that

$$\delta[a(x-X)] = \frac{1}{|a|}\delta(x-X).$$