

Some useful Maths.

Scalar triple product

$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$$

Note the 'order' is preserved.

The products $\mathbf{A} \bullet (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \bullet (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \bullet (\mathbf{B} \times \mathbf{A})$ are of opposite sign.

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \bullet \mathbf{C}) - \mathbf{C} (\mathbf{A} \bullet \mathbf{B})$$

Gradient

Suppose we have a function of three variables, the temperature, T , in a room for example, then there are different temperature gradients in different directions. Thus it would appear that there are an infinity of possible spatial differentials of T . However T can only be a single value at any point x, y, z , thus its total derivative dT is given by:

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz$$

$$\text{so } dT = \left(\left(\frac{\partial T}{\partial x} \right) \hat{x} + \left(\frac{\partial T}{\partial y} \right) \hat{y} + \left(\frac{\partial T}{\partial z} \right) \hat{z} \right) \bullet (dx \hat{x} + dy \hat{y} + dz \hat{z}) = \nabla T \bullet dl$$

where

$$\nabla T \equiv \left(\frac{\partial T}{\partial x} \right) \hat{x} + \left(\frac{\partial T}{\partial y} \right) \hat{y} + \left(\frac{\partial T}{\partial z} \right) \hat{z}$$

is the gradient of T

$$dT = \nabla T \bullet dl = |\nabla T| |dl| \cos \theta$$

where θ is the angle between ∇T and dl .

If now we fix the magnitude of dl and search for the maximum of dT we find it occurs when $\theta = 0$. So greatest dT occurs when I move in the same direction as ∇T .

Thus the gradient ∇T points in the direction of maximum rate of change (in space) of the function T .

The magnitude $|\nabla T|$ gives the rate of change along this maximal direction.

The vector operator ∇ may operate in three different ways.

Firstly it may act on a scalar quantity, for example temperature or pressure, when it gives the gradient of the scalar, e.g. ∇T . This quantity is of course itself a vector and is called the **gradient** of T .

Secondly it may act on a vector quantity, for example the electric field \mathbf{E} , via the dot product, e.g. $\nabla \cdot \mathbf{E}$. This quantity is of course a scalar and is called the **divergence** of \mathbf{E} .

Thirdly it may act on a vector quantity, for example the magnetic field \mathbf{B} , via the cross product, e.g. $\nabla \times \mathbf{B}$. This quantity is of course a vector and is called the **curl** of \mathbf{B} .

Some of the rules for using ∇ :

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second derivatives:

$$\nabla^2 = \nabla \cdot (\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \text{ is the Laplacian of } T$$

$$\nabla \times (\nabla T) = 0 \text{ always for any scalar}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0 \text{ always for any vector } \mathbf{v}.$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

Gauss's theorem, Green's theorem or the Divergence Theorem:

$$\int_V (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

Stokes' theorem: $\oint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$ (Surface S bounded by P)