

# Matrix algebra

$$\mathbf{A} = (a_{jk}) = \begin{matrix} & \begin{matrix} \text{Column} \\ \downarrow \end{matrix} & & & \\ \begin{matrix} \left( \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix} \right) & \leftarrow & \text{Row} & & \end{matrix}$$

## Row Vector:

$$\underline{\mathbf{A}} = (a_{11} \ a_{12} \ \dots \ a_{1n})$$

## Column vector:

$$\underline{\mathbf{B}} = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}$$

## Square matrix:

$$c = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & \cdot \\ \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \\ c_{n1} & \cdot & \dots & c_{nn} \end{pmatrix}$$

$$\text{“TRACE”} = \sum_{j=k} c_{jk}$$

## Symmetric matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

$a_{23}$  points to the element 4 in the second row, third column.

$a_{32}$  points to the element 4 in the third row, second column.

## Skew-symmetric:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$$

## Diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & a_{nn} \end{pmatrix}$$

$$Ex : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

## Unit matrix

“ I ”

$$\mathbf{I} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}$$

## Zero matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

# Operations

## Equality

For two matrices of the same order:  $\mathbf{A} = \mathbf{B}$  if and only if

$$a_{jk} = b_{jk}$$

Ex: 
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} a & 3-a \\ 3 & 2+2_a \end{pmatrix}$$

## Multiplication by a number

$$\mathbf{A} = (a_{jk})$$

$\lambda$  – number

$$\lambda \mathbf{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ \lambda a_{m1} & \cdot & \dots & \lambda a_{mn} \end{pmatrix}$$

Ex: 
$$2 \cdot \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 8 & 2 \\ -4 & 0 \end{pmatrix}$$

## Addition “A+B”

$$\mathbf{A+B} = (a_{jk} + b_{jk}) \Leftrightarrow \begin{pmatrix} a_{11} + b_{11} & \cdot & \cdot & a_{1n} + b_{1n} \\ \cdot & & & \\ \cdot & & & \\ a_{m1} + b_{m1} & \cdot & \cdot & a_{mn} + b_{mn} \end{pmatrix}$$

i)  $\mathbf{A+B = B+C}$

ii)  $\mathbf{A+(B+C) = (A+B)+C}$

*Ex* :  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$

$\swarrow \quad \downarrow \quad \nwarrow$   
 $3 + (-1) = 2$

## Subtraction “A - B”

$$\mathbf{A - B} = (a_{jk} - b_{jk})$$

# Multiplication of Matrices

“AB”

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdot & \cdot & a_{1n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{m1} & \cdot & \cdot & a_{mn} \end{pmatrix}$$

n

$$\mathbf{B} = \begin{pmatrix} b_{11} & \cdot & \cdot & b_{1p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ b_{n1} & \cdot & \cdot & a_{mn} \end{pmatrix}$$

p

$$\mathbf{AB} = \begin{pmatrix} b_{11} & \cdot & \cdot & \cdot & \cdot & b_{n1} \\ * & + & \dots & \dots & + & * \\ a_{11} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \cdot & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdot & \cdot & \cdot & \cdot & b_{1p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & \cdot & \cdot & \cdot & \cdot & b_{np} \end{pmatrix} =$$

$$= \begin{pmatrix} \underbrace{a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}}_{c_{11}} & \dots & \underbrace{a_{11}b_{1p} + \dots + a_{1n}b_{np}}_{c_{1p}} \\ \dots & \dots & \dots \\ \underbrace{a_{m1}b_{11} + a_{m2}b_{21} + \dots + a_{mn}b_{n1}}_{c_{m1}} & \dots & \underbrace{a_{m1}b_{1p} + \dots + a_{mn}b_{np}}_{c_{mp}} \end{pmatrix}$$

$$\mathbf{AB}=\mathbf{C}$$

$$\mathbf{C}=(\mathbf{C}_{jk})$$

Ex:

$$\mathbf{A}=\begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix} \quad \mathbf{B}=\begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 \cdot 3 + 1 \cdot 2 + 4 \cdot 4, & 2 \cdot 5 + 1 \cdot (-1) + 4 \cdot 2 \\ (-3) \cdot 3 + 0 \cdot 2 + 2 \cdot 4, & (-3) \cdot 5 + 0 \cdot (-1) + 2 \cdot 2 \end{pmatrix} = \\ &= \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix} \end{aligned}$$

$$\begin{array}{c} \mathbf{A} \quad \cdot \quad \mathbf{B} \quad = \quad \mathbf{C} \\ (m \times n)(n \times p) = (m \times p) \end{array}$$

The diagram illustrates the compatibility of matrix dimensions for multiplication. It shows the equation  $(m \times n)(n \times p) = (m \times p)$ . A curved arrow points from the  $n$  in the first matrix to the  $n$  in the second matrix, indicating that the number of columns of the first matrix must equal the number of rows of the second matrix. Another curved arrow points from the  $m$  in the first matrix to the  $m$  in the result matrix, and a third curved arrow points from the  $p$  in the second matrix to the  $p$  in the result matrix, indicating that the number of rows of the first matrix and the number of columns of the second matrix determine the dimensions of the product matrix.

## Properties of multiplication

!

$$AB \neq BA$$

$$\mathbf{A(BC) = (AB)}$$

$$\mathbf{A(B + C) = AB + AC}$$

$$\mathbf{(B + C)A = BA + CA}$$


Be careful  
with the  
sides

$$\mathbf{A \cdot I = I \cdot A = A}$$

$$\mathbf{A \cdot 0 = 0}$$

# Transpose of a Matrix    " $\mathbf{A}^T$ "

$$\mathbf{A} = (a_{jk})$$

$$\mathbf{A}^T = (a_{kj})$$


Ex:  $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$

$$\mathbf{A}^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

$$\mathbf{A}^+ \equiv \mathbf{A}^T$$

Properties     $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

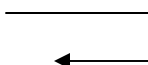
$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$\begin{pmatrix} 2 & -4 \\ -4 & 3 \end{pmatrix}$$

$$\mathbf{A}^T = \mathbf{A}$$

$$\mathbf{A}^T = -\mathbf{A}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 4 \\ -2 & -4 & 2 \end{pmatrix}$$


## Multiplication of vectors     "X · R"

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\underline{\mathbf{X}} \cdot \underline{\mathbf{R}} = |\underline{X}| \cdot |\underline{R}| \cdot \cos \Theta = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$$\underline{\mathbf{X}}^T \cdot \underline{\mathbf{R}} = (x_1 \quad x_2 \quad x_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = x_1 v_1 + x_2 v_2 + x_3 v_3$$

Ex:      $\mathbf{A} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \overline{\mathbf{A}} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

b)

If  $\mathbf{A} = \overline{\mathbf{A}}^T$

$$\mathbf{A} = -\overline{\mathbf{A}}^T$$

Ex:

$$\mathbf{A} = \begin{pmatrix} 5 & -2i \\ 2i & 3 \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} 5 & 2i \\ -2i & 3 \end{pmatrix} \quad \overline{\mathbf{A}}^T = \begin{pmatrix} 5 & -2i \\ 2i & 3 \end{pmatrix}$$

## Determinants

$$\text{Def } (\mathbf{A}) \equiv \Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{n1} & \cdot & \dots & a_{nn} \end{vmatrix}$$

### How to find det(A)

#### Minor

$$\mathbf{A} = \begin{pmatrix} \boxed{a_{11}} & \vdots & \boxed{a_{1n}} \\ \dots & \boxed{a_{jk}} & \dots \\ \boxed{a_{n1}} & \vdots & \boxed{a_{nn}} \end{pmatrix}$$

$$\text{Minor of } a_{jk} = \begin{vmatrix} \boxed{a_{11}} & \boxed{a_{1n}} \\ \boxed{a_{n1}} & \boxed{a_{nn}} \end{vmatrix}$$

Ex:  $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{pmatrix}$  Minor of  $a_{32} = \begin{vmatrix} 2 & 1 & 3 \\ -3 & 5 & 0 \\ 4 & 3 & 1 \end{vmatrix}$

# Cofactor

$$A_{jk} \equiv \text{Cofactor of } a_{jk} = \text{minor of } a_{jk} \cdot (-1)^{j+k}$$

Ex: Cofactor of  $a_{32} = (-1)^{3+2}$

$$A_{jk} = - \begin{vmatrix} 2 & 1 & 3 \\ -3 & 5 & 0 \\ 4 & 3 & 1 \end{vmatrix}$$

$$\text{Det}(A) = \sum_{k=1}^n a_{jk} A_{jk}$$

Ex:

$$\det(A) = a_{j1} \cdot A_{j1} + \dots + a_{j8} \cdot A_{j8} + \dots$$

Ex:  $2 \times 2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$3 \times 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot [a_{22} \cdot a_{33} - a_{23} \cdot a_{32}] + (-a_{12}) \cdot [a_{21} \cdot a_{33} - a_{23} \cdot a_{31}] + a_{13} \cdot [a_{21} \cdot a_{32} - a_{22} \cdot a_{31}]$$

Ex: Calculate the determinant of matrix A:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 6 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A}) &= 3 \cdot [(-2) \cdot 4 - 0 \cdot 3] + \\ & (-1) \cdot [0 \cdot 4 - 0 \cdot 3] + \\ & 6 \cdot [0 \cdot 0 - 2 \cdot 0] = -24 \end{aligned}$$

alternatively (using 3<sup>rd</sup> row):

$$\det(\mathbf{A}) = 4 \cdot [3 \cdot (-2) - 0 \cdot 1] = -24$$

## Theorems (15-7)

If we multiply the elements of any row (or column) by a given number and add to corresponding elements of any other row (or column), then the value of the determinant remains the same.

$$\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} x_1 + ky_1 & x_2 + ky_2 & x_3 + ky_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

$$\det(\mathbf{A}) = \det(\mathbf{B})$$

### Th: 15-5

If any two rows (or columns) are the same or proportional, the determinant is zero.

$$\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 \\ kx_1 & kx_2 & kx_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \leftarrow \begin{array}{l} \text{multiply by } \left(\frac{1}{k}\right) \text{ and add to} \\ \text{the first row} \end{array}$$

$$\mathbf{B} = \begin{pmatrix} \left(x_1 - \frac{k}{k}x_1\right) & \left(x_2 - \frac{k}{k}x_2\right) & \left(x_3 - \frac{k}{k}x_3\right) \\ kx_1 & kx_2 & kx_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ kx_1 & kx_2 & kx_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

$\det(\mathbf{A}) = \det(\mathbf{B})$  using theorem 15-7

$$\det(\mathbf{B}) = 0 = \begin{vmatrix} 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

← All zeros in a row



$$\det(\mathbf{A}) = 0$$

$$\mathbf{A} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_2 & y_3 & z_3 \end{pmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$   
 $\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}$   
 vectors

Th. 15-10:

If we can find such  $\lambda_1, \lambda_2, \lambda_3$   
 so that not all of them = 0

And:  $\lambda_1 \underline{x} + \lambda_2 \underline{y} + \lambda_3 \underline{z} = \underline{0} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [*]$

Then  $\det(\mathbf{A}) = 0$

- $\underline{x}, \underline{y}, \underline{z} \longrightarrow$

- $\mathbf{A} \longrightarrow$

(otherwise)

- $\underline{x}, \underline{y}, \underline{z} \longrightarrow$

- $\mathbf{A} \longrightarrow$

**Inverse matrix:**      " $\mathbf{A}^{-1}$ "

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{I}$$

Then B is called inverse and denoted  $\mathbf{A}^{-1}$

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$$

Theorem (15-11)      If  $\mathbf{A}$ -nonsingular (i.e.  $\det(\mathbf{A}) \neq 0$ )

Then there is a unique inverse matrix  $\mathbf{A}^{-1}$  which can be expressed (found) in the following way:

$$\mathbf{A}^{-1} = \frac{(\mathbf{A}_{jk})^T}{\det(\mathbf{A})}$$

$$1. (\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1} \quad 2. (\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

3. Real numbers     $\mathbf{A}^T = \mathbf{A}^{-1}$  :  $\mathbf{A}$  is

4. Complex numbers  $\mathbf{A}^{-T} = \mathbf{A}^{-1}$ :  $\mathbf{A}$  is

## Example

$$\mathbf{C} = \begin{pmatrix} 3 & 1 & 6 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{Calculated earlier: } \det(\mathbf{C}) = -24$$

$$\text{Find } \mathbf{C}^{-1}. \text{ Matrix of cofactors: } (C_{jk}) = \begin{pmatrix} -8 & 0 & 0 \\ -4 & 12 & 0 \\ 15 & -9 & -6 \end{pmatrix}$$

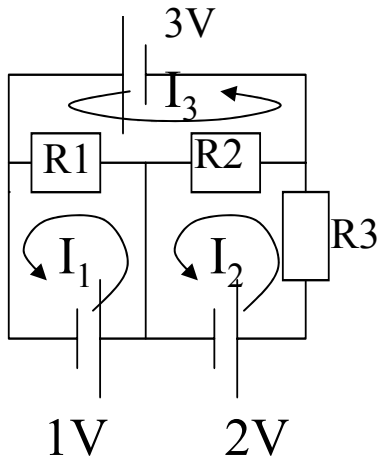
$$(C_{jk}) = \begin{pmatrix} -8 & -4 & 15 \\ 0 & 12 & -9 \\ 0 & 0 & -6 \end{pmatrix}$$

$$\mathbf{C}^{-1} = \frac{(C_{jk})^T}{\det(\mathbf{C})} = -\frac{1}{24} \cdot \begin{pmatrix} -8 & -4 & 15 \\ 0 & 12 & -9 \\ 0 & 0 & -6 \end{pmatrix}$$

Verification:  $\mathbf{C} \mathbf{C}^{-1} = \mathbf{I}$

$$\begin{aligned} & \begin{pmatrix} 3 & 1 & 6 \\ 0 & -2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \cdot \left(\frac{1}{24}\right) \begin{pmatrix} -8 & -4 & 15 \\ 0 & 12 & -9 \\ 0 & 0 & -6 \end{pmatrix} = \\ & -\frac{1}{24} \begin{pmatrix} -8 \cdot 3 + 1 \cdot 0 + 6 \cdot 0 & -4 \cdot 3 + 12 \cdot 1 + 0 \cdot 6 & 15 \cdot 3 - 9 \cdot 1 - 6 \cdot 6 \\ -8 \cdot 0 + (-2) \cdot 0 + 3 \cdot 0 & -4 \cdot 0 + 12 \cdot (-2) + 0 \cdot 0 & 15 \cdot 0 - 9 \cdot (-2) - 6 \cdot 3 \\ -8 \cdot 0 + 0 \cdot 0 + 0 \cdot 4 & -4 \cdot 0 + 12 \cdot 0 + 0 \cdot 4 & 15 \cdot 0 - 9 \cdot 0 - 6 \cdot 4 \end{pmatrix} \\ & = -\frac{1}{24} \begin{pmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

# System of linear Equations



$$\begin{cases} 1 \cdot I_1 - 1 \cdot I_3 = 1 \\ 3 \cdot I_2 - 2 \cdot I_3 = 2 \\ -1 \cdot I_1 - 2 \cdot I_2 + 3 \cdot I_3 = 3 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{\mathbf{A}}\underline{\mathbf{X}}=\underline{\mathbf{R}}$$

$\downarrow$                        $\downarrow$                        $\downarrow$   
 $\underline{\mathbf{A}}$                        $\underline{\mathbf{X}}$                        $\underline{\mathbf{R}}$

How to find  $\underline{\mathbf{X}}$  :

$$\underline{\mathbf{A}}\underline{\mathbf{X}}=\underline{\mathbf{R}}$$

$$\underline{\mathbf{A}}^{-1}\underline{\mathbf{A}}\underline{\mathbf{X}}=\underline{\mathbf{A}}^{-1}\underline{\mathbf{R}} \rightarrow$$

$$\underline{\mathbf{I}}\underline{\mathbf{X}}=\underline{\mathbf{A}}^{-1}\underline{\mathbf{R}} \rightarrow$$

$$\underline{\mathbf{X}}=\underline{\mathbf{A}}^{-1}\underline{\mathbf{R}}$$

$$A^{-1} = \begin{pmatrix} 5/2 & 1 & 3/2 \\ 1 & 1 & 1 \\ 3/2 & 1 & 3/2 \end{pmatrix} \quad \underline{X} = A^{-1} \underline{R}$$

$$X = \begin{pmatrix} 5/2 & 1 & 3/2 \\ 1 & 1 & 1 \\ 3/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 5/2 + 2 \cdot 1 + 3 \cdot 3/2 \\ 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 1 \cdot 3/2 + 2 \cdot 1 + 3 \cdot 3/2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$$

$$X = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix} \quad \longrightarrow \quad \begin{array}{l} I_1 = 9A \\ I_2 = 6A \\ I_3 = 8A \end{array}$$

**Possible solutions:**

$$Det \neq 0, R \neq 0$$

$$Det \neq 0, R = 0$$

$$\underline{X} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$Det = 0, R = 0$$

$$Det = 0, R \neq 0$$

\*\*see page p348, Cramer's rule.


# Eigenvalue problem

A – square matrix of order 3.

$$\underline{X} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$A\underline{X} = \underline{X}'$$





$$\underline{A}\underline{X} = \lambda\underline{X} \quad \leftarrow \text{Eigenvalue problem}$$

↓  
To find

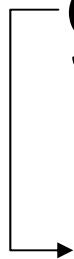
{  
eigenvectors  
eigenvalues

## Eigenvalues

$$\underline{\mathbf{A}}\underline{\mathbf{X}} = \lambda\underline{\mathbf{X}}$$

$$\underline{\mathbf{A}}\underline{\mathbf{X}} - \lambda\underline{\mathbf{X}} = \underline{\mathbf{0}} \quad \text{using distributive law}$$

$$\underbrace{(\underline{\mathbf{A}} - \lambda\underline{\mathbf{I}})}\underline{\mathbf{X}} = \underline{\mathbf{0}}$$



There are two cases for solutions:

1  $\text{Det} \neq 0, R = 0$

$$\underline{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

2  $\text{Det} = 0, R = 0$



Just what we want!





Secular determinant:

$$\begin{vmatrix} (2-\lambda) & 2 \\ -1 & (5-\lambda) \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 2(-1) = 10 - 5\lambda - 2\lambda + \lambda^2 + 2 =$$

$$\underbrace{\lambda^2 - 7\lambda + 12 = 0}$$

$$\lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0$$

$$\lambda_{1,2} = 3, 4 \longrightarrow$$

Conclusion:

## Eigenvectors

$$\begin{pmatrix} 2-3 & 2 \\ -1 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \begin{cases} -1 \cdot x_1 + 2 \cdot x_2 = 0 \\ -1 \cdot x_1 + 2 \cdot x_2 = 0 \end{cases}$$

$x_1 = 2x_2 \leftarrow$



Let  $x_2=1$ , then  $x_1=2$

$$\underline{\mathbf{X}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow$$



$$\underline{\mathbf{Y}} = 3\underline{\mathbf{X}} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \longrightarrow \text{eigenvector of } \mathbf{A} \\ \text{(corresponding to } \lambda=3)$$



$$\begin{cases} -2x_1 + 2x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \quad \text{Let } x_2=1, \text{ then } x_1=1 \quad \underline{\mathbf{X}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Final result:

Eigenvalues  $\lambda_1=3, \lambda_2=4$

Eigenvectors  $\underline{\mathbf{X}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \underline{\mathbf{X}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

### Unit eigenvectors

$$\underline{\mathbf{U}} = \frac{\underline{\mathbf{X}}}{|\underline{\mathbf{X}}|} \quad |\underline{\mathbf{X}}| = \sqrt{x_1^2 + x_2^2 \cdots x_n^2}$$

$$\text{Ex: } \underline{\mathbf{X}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad |\underline{\mathbf{X}}_1| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\Rightarrow \underline{\mathbf{U}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

## Properties of eigenvectors

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

then  $\mathbf{B}^{-1}\mathbf{A}\mathbf{B} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \mathbf{0} & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}$  the diagonal form of  $\mathbf{A}$

$\mathbf{B}^{-1}\mathbf{A}\mathbf{B} \leftarrow$  the transform of  $\mathbf{A}$  by  $\mathbf{B}$



## Canonical form

Theorem 15-13



Eigenvectors are orthogonal

e.g.

$$\begin{aligned}\underline{\mathbf{X}} \perp \underline{\mathbf{Y}} &\Rightarrow \underline{\mathbf{X}}^+ \underline{\mathbf{Y}} = (x_1 \quad x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \\ &= x_1 y_1 + x_2 y_2 = 0\end{aligned}$$

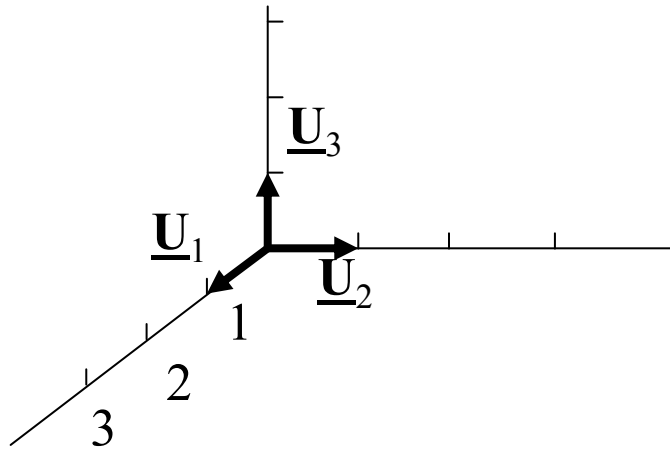


## Basis vectors

$$\underline{\mathbf{U}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\mathbf{U}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{\mathbf{U}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\underline{\mathbf{U}}_1: \quad \mathbf{A}' \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

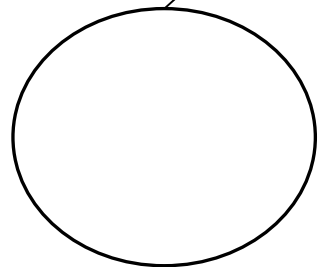
$$\underline{\mathbf{U}}_2: \quad \mathbf{A}' \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$



$$\begin{aligned}
 \underline{\mathbf{U}}_1: \quad & 1 \cdot a'_{11} + 0 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = \lambda_1 \\
 & 1 \cdot a'_{11} + 0 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = 0 \\
 & 1 \cdot a'_{11} + 0 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = 0 \\
 & \dots\dots\dots
 \end{aligned}$$

$$\begin{aligned}
 \underline{\mathbf{U}}_2: \quad & 0 \cdot a'_{11} + 1 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = 0 \\
 & 0 \cdot a'_{11} + 1 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = \lambda_2 \\
 & 0 \cdot a'_{11} + 1 \cdot a'_{12} + 0 \cdot a'_{13} + \cdots + 0 \cdot a'_{1n} = 0 \\
 & \dots\dots\dots
 \end{aligned}$$

$$\mathbf{A}' = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}$$



$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \underline{\mathbf{U}} = \lambda \underline{\mathbf{U}}$$

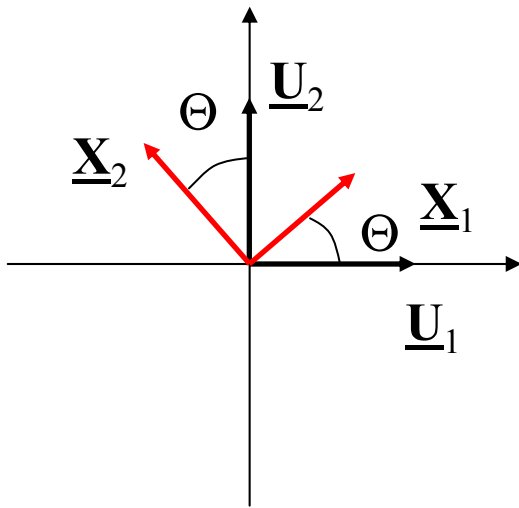
where  $\underline{\mathbf{U}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$\uparrow \quad \uparrow \quad \dots \quad \uparrow$   
 $\underline{\mathbf{U}}_1, \underline{\mathbf{U}}_2, \dots, \underline{\mathbf{U}}_n$

and  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$

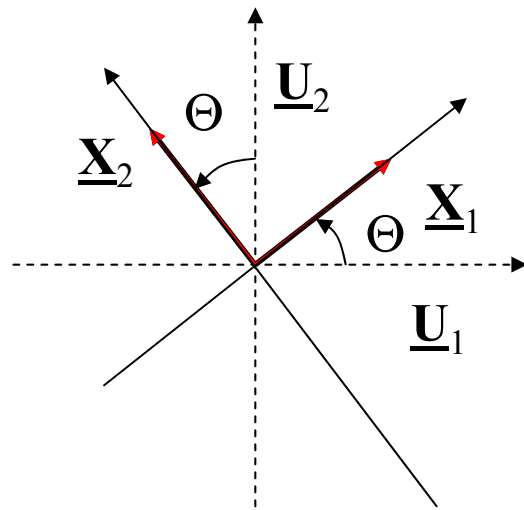
$$\underline{\mathbf{A}}\underline{\mathbf{X}} = \lambda \underline{\mathbf{X}}$$

Ex:



$$\underline{\mathbf{X}}_1 = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad |\underline{\mathbf{X}}_1| = 1$$

$$\underline{\mathbf{X}}_2 = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad |\underline{\mathbf{X}}_2| = 1$$



$$\left. \begin{aligned} \underline{\mathbf{X}}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \underline{\mathbf{U}}_1 \\ \underline{\mathbf{X}}_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \underline{\mathbf{U}}_2 \end{aligned} \right\}$$



Answer:

$$\mathbf{B}^{-1} \mathbf{A} \mathbf{B} = \mathbf{A}'$$

Answer:

$$\mathbf{B}^T \underline{\mathbf{X}} = \underline{\mathbf{U}}$$

\*\* ) eigenvectors should be of unit length

Ex:

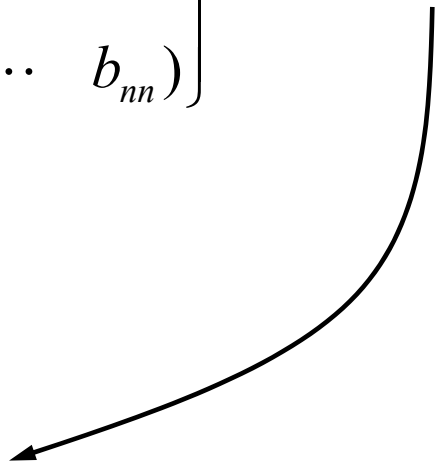
$$\underline{\mathbf{U}}_1: \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is only possible if :

$$\left. \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{pmatrix} \right\} \begin{array}{l} \text{orthogonal} \\ \text{unit eigenvectors} \end{array}$$



$\mathbf{B}^T$





$$(\mathbf{A}' - \lambda \mathbf{I}) \underline{\mathbf{U}} = \underline{\mathbf{0}}$$

$$(\mathbf{B}^{-1} \mathbf{A} \mathbf{B} - \lambda \mathbf{I}) \mathbf{B}^T \underline{\mathbf{X}} = \underline{\mathbf{0}}$$

$$\mathbf{B}^{-1} \mathbf{A} \mathbf{B} \cdot \mathbf{B}^T \underline{\mathbf{X}} - \lambda \mathbf{I} \mathbf{B}^T \underline{\mathbf{X}} = \underline{\mathbf{0}}$$

$$\therefore \mathbf{B}^T = \mathbf{B}^{-1} \Rightarrow \mathbf{B}^{-1} \mathbf{A} \mathbf{B} \cdot \underbrace{\mathbf{B}^{-1} \mathbf{B}^T}_{\mathbf{I}} \underline{\mathbf{X}} - \lambda \mathbf{I} \mathbf{B}^{-1} \underline{\mathbf{X}} = \underline{\mathbf{0}}$$

$$\mathbf{B}^{-1} \mathbf{A} \underline{\mathbf{X}} - \lambda \mathbf{I} \mathbf{B}^{-1} \underline{\mathbf{X}} = \underline{\mathbf{0}}$$

$$\mathbf{B}^{-1} (\underbrace{\mathbf{A} \underline{\mathbf{X}} - \lambda \mathbf{I} \underline{\mathbf{X}}}_{= \mathbf{0}}) = \underline{\mathbf{0}}$$



transformations are valid

## Summary

If  $\mathbf{A}$  – Hermitian

Eigenvalue problem  $\mathbf{A}\underline{\mathbf{X}} = \lambda\underline{\mathbf{X}}$

can be transformed to a canonical form in the basis of unit eigenvectors:

$$\mathbf{A}'\underline{\mathbf{U}} = \lambda\underline{\mathbf{U}}$$

where

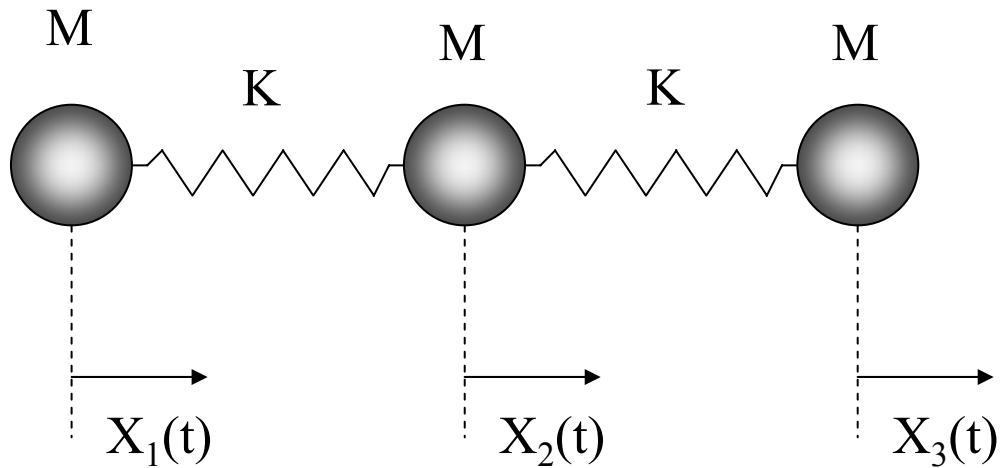
$$\mathbf{A}' = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$$

and

$$\underline{\mathbf{U}} = \mathbf{B}^T \underline{\mathbf{X}}$$

where  $\mathbf{B}$  is an orthogonal matrix of unit eigenvectors (arranged as columns)

# Eigenvalue problem (practical example)



Periodic solution:

$$x_k(t) = x_k e^{i\omega t}$$

$$k = 1, 2, 3$$

After substitution:

$$-M\omega x_1 = -Kx_1 + Kx_2$$

$$-M\omega x_2 = Kx_1 - 2Kx_2 - Kx_3$$

$$-M\omega x_3 = Kx_2 - Kx_3$$

In matrix form:


$$\rightarrow \omega^2 \underline{\mathbf{X}} = \mathbf{A} \underline{\mathbf{X}} \quad \text{where} \quad A = -\frac{K}{M} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

or equivalently:

$$\rightarrow (\mathbf{A} - \omega^2 \mathbf{I}) \underline{\mathbf{X}} = 0$$

Eigenvalue problem !

The solutions ( i.e. vector  $\underline{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$  )

Let us first redefine  $\mathbf{A}$   
assuming that 

$$\frac{K}{M} \equiv 1 \quad \lambda \equiv \omega^2$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Expected eigenmodes

$$\omega = \sqrt{\lambda \frac{K}{M}}$$

## Secular determinant

$$\begin{array}{l} R_1 \\ R_3 \end{array} \left| \begin{array}{ccc} (1-\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{array} \right| = 0$$

$$\begin{array}{c} C_1 \\ C_3 \end{array} \left| \begin{array}{ccc} (1-\lambda) & 0 & -(1-\lambda) \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{array} \right| \leftarrow R_1 - R_3 = 0$$

$$\begin{array}{c} C_1 + C_3 \end{array} \left| \begin{array}{ccc} (1-\lambda) & 0 & 0 \\ -1 & (2-\lambda) & -2 \\ 0 & -1 & (1-\lambda) \end{array} \right| = 0$$

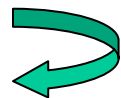
Characteristic equation:

$$(1-\lambda)[(2-\lambda)(1-\lambda)-2] = 0$$

$$(1-\lambda)[\lambda^2 - 3\lambda] = \lambda(1-\lambda)(\lambda-3) = 0$$

Eigenvalues:  $\lambda_{1,2,3} = 0, 1, 3$

Resonance frequencies:  $\omega_{1,2,3} = 0, \sqrt{\frac{K}{M}}, \sqrt{\frac{3K}{M}}$



## Eigenvectors

$$\lambda_1 = 0: \begin{cases} x_1 - x_2 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \quad \begin{array}{l} \text{Let } x_2 = 1 \\ \text{From 1}^{\text{st}}: x_1 = x_2 = 1 \\ \text{From 3}^{\text{rd}}: x_3 = x_2 = 1 \end{array}$$



$$\underline{\mathbf{X}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{Unit vector:} \quad \underline{\mathbf{U}}_1 = \frac{\underline{\mathbf{X}}_1}{|\underline{\mathbf{X}}_1|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1: \begin{cases} -x_2 = 0 \\ -x_1 + x_2 - x_3 = 0 \\ -x_2 = 0 \end{cases} \quad \begin{array}{l} x_2 = 0 \\ x_1 = -x_3, \text{ Let } x_1 = 1 \\ x_3 = -1 \end{array}$$



$$\underline{\mathbf{X}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{Unit vector:} \quad \underline{\mathbf{U}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 0: \begin{cases} -2x_1 - x_2 = 0 \\ -x_1 - x_2 - x_3 = 0 \\ -x_2 - 2x_3 = 0 \end{cases} \quad \begin{array}{l} \text{Let } x_2 = 2 \\ x_1 = -1 \\ x_3 = -1 \end{array}$$

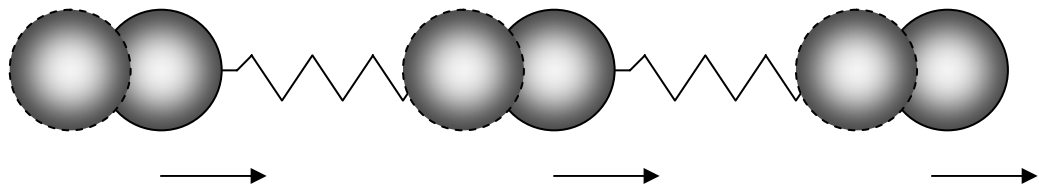


$$\underline{\mathbf{X}}_3 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \text{Unit vector:} \quad \underline{\mathbf{U}}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

### Modes of oscillation

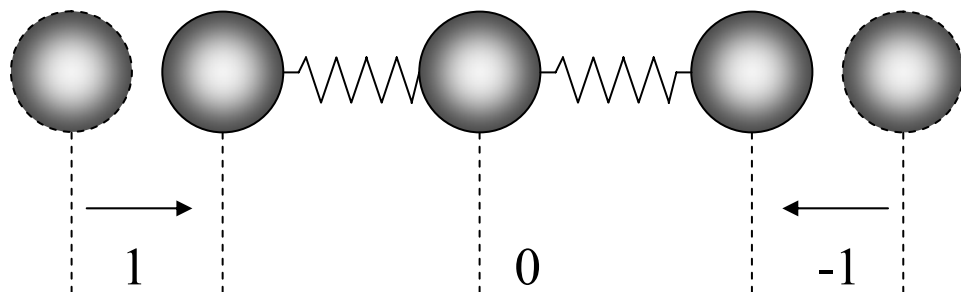
$\omega = 0$ , No oscillation, uniform movement

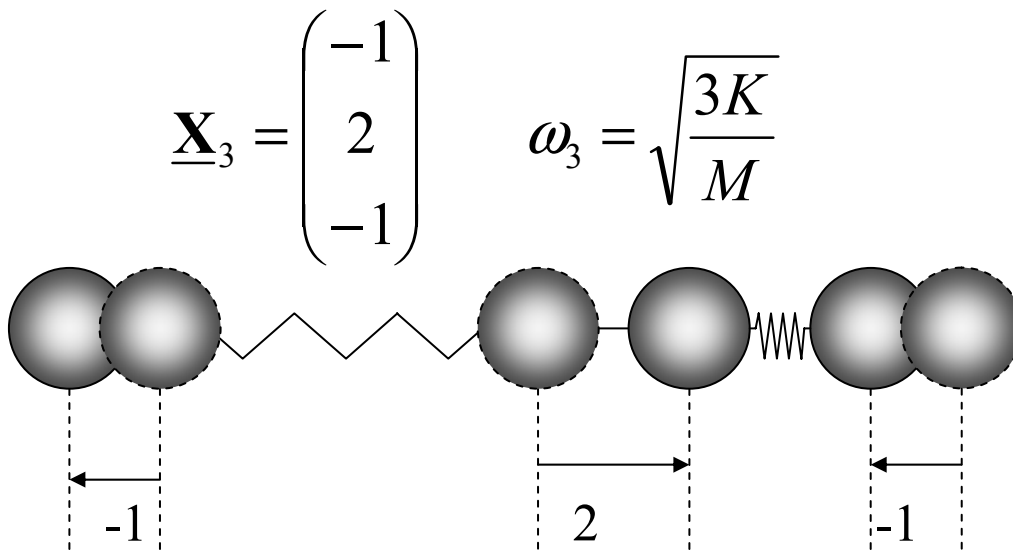
$$\underline{\mathbf{X}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



$$\omega_2 = \sqrt{\frac{K}{M}}$$

$$\underline{\mathbf{X}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$





The eigenvectors represent the **states** of the system corresponding to different **modes** defined by the eigenvalues

**Transformation to the eigenvector basis**

$$\underline{\mathbf{X}}' = \mathbf{B}^T \underline{\mathbf{X}}$$

Transformation requires that:

B - orthogonal

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\mathbf{A} = \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}$$

$$\left\{ \begin{array}{l} \underline{\mathbf{X}}_1 \cdot \underline{\mathbf{X}}_2 = (1 \cdot (-1) + 0 \cdot 1 + 1 \cdot 1) = 0 \\ \underline{\mathbf{X}}_1 \cdot \underline{\mathbf{X}}_3 = (1 \cdot (-1) + 2 \cdot 1 + 1 \cdot (-1)) = 0 \\ \underline{\mathbf{X}}_2 \cdot \underline{\mathbf{X}}_3 = (1 \cdot (-1) + 0 \cdot 2 + (-1) \cdot (-1)) = 0 \end{array} \right.$$

Orthogonal  $\Leftrightarrow$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \underline{\mathbf{X}}' = \lambda \underline{\mathbf{X}}'$$

Where 

$$\lambda = 0, \frac{K}{M}, \frac{3K}{M} \quad \text{and} \quad \underline{\mathbf{X}}'_{1,2,3} = \begin{pmatrix} x'_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x'_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x'_3 \end{pmatrix}$$

Accordingly:

$$\begin{cases} -M\omega^2 x'_1 = 0 + 0 + 0 \\ -M\omega^2 x'_2 = 0 - Kx'_2 + 0 \\ -M\omega^2 x'_3 = 0 + 0 - 3Kx'_3 \end{cases}$$

$$\underline{\mathbf{X}}' = \mathbf{B}^T \underline{\mathbf{X}} \quad \Leftrightarrow \quad \begin{cases} x'_1 = x_1 + x_2 + x_3 \\ x'_2 = x_1 - x_3 \\ x'_3 = -x_1 + 2x_2 - x_3 \end{cases}$$



## Summary

(coupled linear differential equation)

Given: A physical system of order  $n$ , which can be described by eigenvalue problem:

$$\underline{\mathbf{A}}\underline{\mathbf{X}} = \lambda\underline{\mathbf{X}}$$

In which we are looking for:

eigenvalues  $\rightarrow$  **modes**

eigenvectors  $\rightarrow$  **states**

Can be diagonalised by transforming to the basis of the eigenvectors

This can only be done if A is symmetric

In the eigenvector basis matrix A has diagonal form where the main diagonal consists of eigenvalues

## Degeneracy

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \longleftarrow$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 = -(\lambda - 1)^2(\lambda - 4) = 0$$

$$\lambda_{1,2,3} = 1, 1, 4$$

$\lambda=1$  is used twice!

↑ degenerate

Eigenvectors:

$$\lambda_1 = 1: \quad \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \quad \begin{array}{l} \text{Let } x_2 = -2 \\ x_1 = 2 - x_3 \\ x_1 = x_3 = 1 \end{array}$$

$$\underline{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



$$\underline{\mathbf{X}}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$



$$\lambda=4: \quad \underline{\mathbf{X}}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

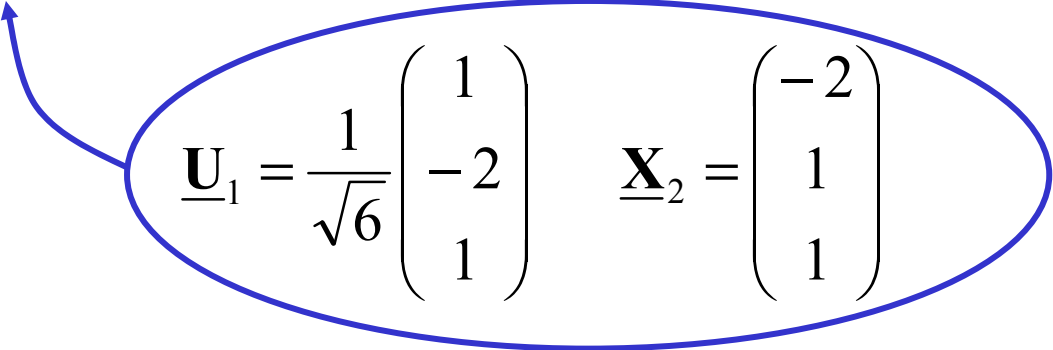
However:  $\underline{\mathbf{X}}_1 \cdot \underline{\mathbf{X}}_2 = -2 \cdot 1 + (-2) \cdot 1 + 1 \cdot 1 = -3 \neq 0$



$$\underline{\mathbf{X}}_1 \cdot \underline{\mathbf{X}}_3 = \underline{\mathbf{X}}_2 \cdot \underline{\mathbf{X}}_3 = 0$$


To make the second vector orthogonal use **Gram-Schmidt** orthonormalisation process.

$$\underline{\mathbf{X}}'_2 = \underline{\mathbf{X}}_2 - (\underline{\mathbf{X}}_2 \cdot \underline{\mathbf{U}}_1) \underline{\mathbf{U}}_1 =$$



$$\underline{\mathbf{U}}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \underline{\mathbf{X}}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \left( \frac{1}{\sqrt{6}} (-2 \cdot 1 - 2 \cdot 1 + 1) \right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}} = \begin{pmatrix} -3/2 \\ 0 \\ 3/2 \end{pmatrix}$$



$$\underline{\mathbf{U}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Check that it satisfies the eigenvalue problem:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \left. \begin{array}{l} -2 + 1 = -1 \\ -1 + 1 = 0 \\ -1 + 2 = 1 \end{array} \right\} \text{OK}$$

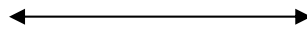
# Differential equation as an eigenvalue problem

So far:

$$\mathbf{A}\underline{\mathbf{X}} = \lambda\underline{\mathbf{X}}$$



$\underline{\mathbf{X}}_n$



$\lambda_n$



$$\underline{\mathbf{X}}_n \cdot \underline{\mathbf{X}}_m = 0 \quad m \neq n$$

$$\underline{\mathbf{X}}_n \cdot \underline{\mathbf{X}}_m \neq 0 \quad m = n$$

can be also normalised so that:

$$\underline{\mathbf{X}}_n \cdot \underline{\mathbf{X}}_n = 1 \quad |\underline{\mathbf{X}}_n|^2 = 1$$



## The General Eigenvalue Problem

$$\mathbf{A}\underline{\Phi} = \lambda\underline{\Phi} \quad \text{where}$$
$$\underline{\Phi} = \sum_n c_n \underline{\mathbf{X}}_n$$
$$\underline{\Phi} = c_1 \underline{\mathbf{X}}_1 + c_2 \underline{\mathbf{X}}_2 + c_3 \underline{\mathbf{X}}_3 \cdots$$

$$\mathbf{A}c_1 \underline{\mathbf{X}}_1 + \mathbf{A}c_2 \underline{\mathbf{X}}_2 + \mathbf{A}c_3 \underline{\mathbf{X}}_3 + \cdots = \lambda c_1 \underline{\mathbf{X}}_1 + \lambda c_2 \underline{\mathbf{X}}_2 + \cdots$$

$$\mathbf{A} \left( \sum_n c_n \underline{\mathbf{X}}_n \right) = \sum_n \lambda_n c_n \underline{\mathbf{X}}_n$$



YES, we can

To start with, introduce Kroenicker delta:

$$\underline{\mathbf{X}}_n \cdot \underline{\mathbf{X}}_m = \delta_{n,m} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$$\underline{\Phi} = \sum_n c_n \underline{\mathbf{X}}_n$$

$$\underline{\Phi} \cdot \underline{\mathbf{X}}_p = \left( \sum_n c_n \underline{\mathbf{X}}_n \right) \underline{\mathbf{X}}_p = \sum_n c_n \underline{\mathbf{X}}_n \underline{\mathbf{X}}_p =$$

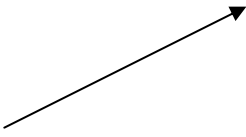
$$\underline{\Phi} \cdot \underline{\mathbf{X}}_p = \sum_n c_n \underline{\mathbf{X}}_n \underline{\mathbf{X}}_p = \sum_n c_n \delta_{n,p} = c_p$$

$$f_n(x) = \frac{1}{\sqrt{2L}} \sin\left(\frac{2\pi nx}{L}\right)$$

$$g_n(x) = \frac{1}{\sqrt{2L}} \cos\left(\frac{2\pi nx}{L}\right)$$

$$\int_{-L}^L f_n(x) f_m(x) dx = \frac{1}{2L} \int_{-L}^L \sin\left(\frac{2\pi nx}{L}\right) \sin\left(\frac{2\pi mx}{L}\right) dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$= \delta_{n,m}$



Also:

$$F(x) = \sum_n c_n f_n(x) \quad \text{-- "sin" Fourier series}$$

For  $c_p$  coefficient:

$$\begin{aligned}\int_{-L}^L F(x) f_p(x) dx &= \sum_n c_n \int_{-L}^L f_n(x) f_p(x) dx = \\ &= \sum_n c_n \delta_{n,p} = c_p\end{aligned}$$

Conclusion:

Restrictions:

## Eigenvalue problem in differential equations

Suppose we have a differential operator  $L(x)$ :

$$L(x) = \frac{A\partial^2}{\partial x^2} + B \frac{\partial}{\partial x} + Cu(x)$$

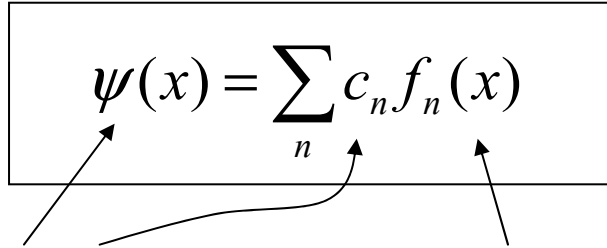
+

$$\psi(x) = \sum_n c_n f_n(x)$$

$$L(x)\psi(x) = \lambda\psi(x)$$

## The Method

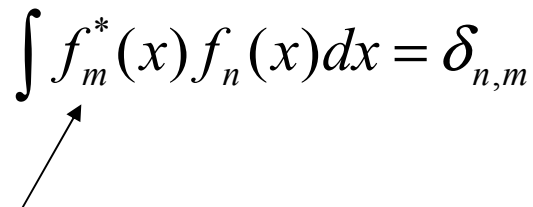
Is to try for a solution for  $\psi(x)$  in the form:

$$\psi(x) = \sum_n c_n f_n(x)$$
A rectangular box contains the equation  $\psi(x) = \sum_n c_n f_n(x)$ . Three arrows point from below to the terms: one to  $\psi(x)$ , one to the summation symbol  $\sum$ , and one to  $f_n(x)$ .

The task -- to find  $c_n$

$$\sum_n L(x) c_n f_n(x) = \sum_n \lambda c_n f_n(x)$$

The trick – use orthonormal properties of  $f_n(x)$  for solving the equation.

$$\int f_m^*(x) f_n(x) dx = \delta_{n,m}$$
An arrow points from the bottom left towards the term  $f_m^*(x)$  in the equation.

$$\sum_n c_n \int \underbrace{f_m^* L(x) f_n(x)}_{L_{mn}} dx = \sum_n \lambda c_n \int \underbrace{f_m^*(x) f_n(x)}_{\delta_{m,n}} dx$$

$$\sum_n c_n L_{mn} = \sum_n \lambda c_n \delta_{m,n}$$

Repeat for All m

$$m=1 \quad L_{11}c_1 + L_{12}c_2 + \dots = 1 \cdot \lambda c_1 + 0 \cdot \lambda c_2 + 0 \cdot \lambda c_3 + \dots$$

$$m=2 \quad L_{21}c_1 + L_{22}c_2 + \dots = 0 + 1 \cdot \lambda c_2 + 0 + \dots$$

$$m=3 \quad L_{31}c_1 + L_{32}c_2 + \dots = 0 + 0 + 1 \cdot \lambda c_3 + \dots$$

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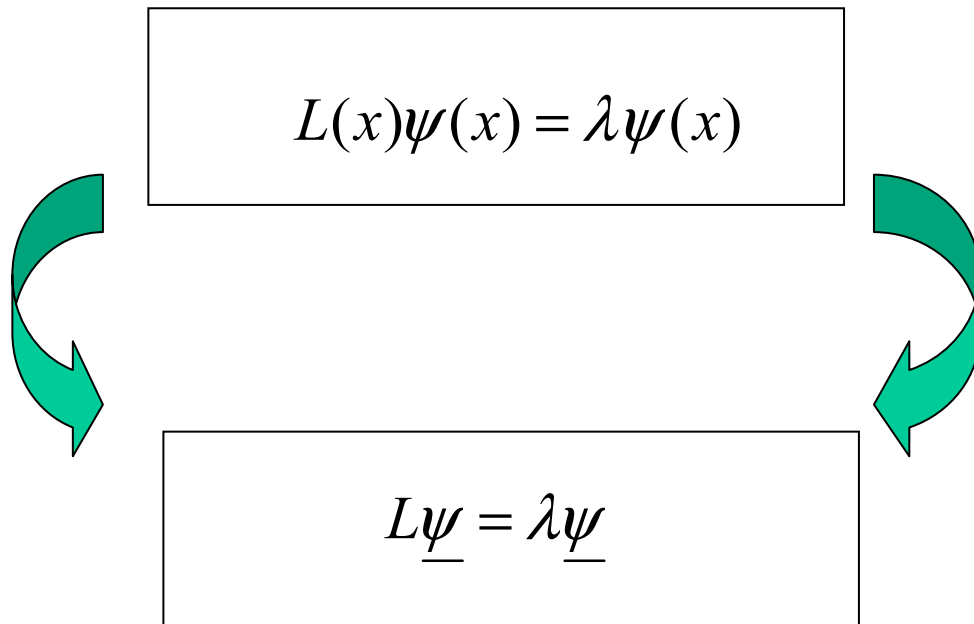
$$L\underline{\psi} = \lambda\underline{\psi}$$



$$L = \begin{pmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ddots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{pmatrix} \quad \underline{\psi} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\lambda = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

**Result:**



## Advantages

## Disadvantages

Ex.:  $\frac{1}{\sqrt{2\pi}} \sin\left(\frac{2\pi nx}{L}\right)$        $N = 1, 2, 3, \dots \text{infinity}$

### Example:

A particle of mass  $M$  moves in the 1D box  $0 < x < L$ . Consider the set of functions  $\{\exp(ikx)\}$  as a basis for the wavefunction of the particle, i.e. as a basis for the solution  $\psi(x)$  to the Schrödinger equation

$$-\frac{\hbar^2}{2M} \frac{d^2}{dx^2} \psi(x) = E \psi(x)$$

- What values must  $k$  take for the boundary condition  $\psi(0) = \psi(L)$  to be satisfied.
- Form an orthonormal basis set  $\{f_n(x)\}$
- Write the Schrödinger equation for the particle as a matrix eigenvalue problem and find the possible values of  $E$

- Next assume that a potential

$$V(x) = V_0 \sin\left(\frac{2\pi x}{L}\right) \quad V_0 = \sqrt{1.24} \frac{2\pi^2 \hbar^2}{ML^2}$$

is switched on inside the box.

- i. Construct the matrix elements associated with  $V(x)$  with respect to the basis set  $\{f_n(x)\}$
- ii. Assume that  $\{f_1, f_2\}$  forms a complete set of the new Hamiltonian:

$$-\frac{\hbar^2}{2M} \frac{d^2}{dx^2} \psi(x) + V(x)$$

Find the two eigenvalues and the corresponding eigenvectors.

## Solution

$$\psi(x) = a \exp(ikx)$$

$$\psi(0) = \psi(L) \quad \exp(ikL) = 1$$

$$kL = 2\pi n$$

$$k = k_n = \frac{2\pi n}{L}$$

$$n = 0, 1, 2, \dots, \infty$$

$$\{f_n(x) = A_n \exp(ik_n x)\}$$

$$\int_{-L}^L f_n(x) f_m(x) dx = A_n^* A_m \int_0^L e^{-ik_n x} e^{ik_m x} dx =$$

$$= A_n^* A_m \int_0^L e^{-i(k_n - k_m)x} dx =$$

$$= A_n^* A_m \left[ \frac{e^{-i(k_n - k_m)x}}{i(k_n - k_m)} \right]_0^L dx =$$

$$= A_n^* A_m \cdot 0 = 0$$

$$\int_0^L f_n^*(x) f_n(x) dx = A_n^* A_n \int_0^L dx = A_n^* A_n L = 1$$

$$\int_0^L f_n^*(x) f_m(x) dx = \delta_{n,m}$$

$$\Psi(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$\underbrace{-\frac{\hbar^2}{2M} \frac{d^2}{dx^2}}_{L(x)} \psi(x) = E \psi(x)$$

$$L(x) \sum_n c_n f_n(x) = E \sum_n c_n f_n(x)$$

$$\int_0^L f_m^* L(x) \sum_n c_n f_n(x) dx = E \int_0^L f_m^*(x) \sum_n c_n f_n(x) dx$$

$$\underbrace{\sum_n c_n \int_0^L f_m^* L(x) f_n(x) dx}_{\text{LHS}} = E \underbrace{\sum_n c_n \int_0^L f_m^*(x) f_n(x) dx}_{\text{RHS}}$$

$$\sum_n c_n L_{mn} = E \sum_n c_n \delta_{m,n}$$

$$\sum_n c_n L_{mn} = E c_m$$

$$\begin{pmatrix} L_{11} & L_{12} & \cdots \\ L_{21} & \ddots & \\ \vdots & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$



$$\begin{pmatrix} \frac{\hbar^2 k_1^2}{2M} & & \mathbf{0} \\ & \frac{\hbar^2 k_2^2}{2M} & \\ \mathbf{0} & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$$E_n = \frac{\hbar^2}{2M} k_n^2 \quad n = 0, 1, 2 \dots \infty$$

$$k_n = \frac{2\pi n}{L}$$

$$E_n = \left( \frac{2\pi^2 \hbar^2}{ML^2} \right) n^2$$

$$\text{ii) } V(x) = V_0 \sin\left(\frac{2\pi x}{L}\right)$$

$$\begin{aligned} V_{mn} &= \int_0^L f_m^* V(x) f_n(x) dx = \\ &= V_0 \frac{1}{L} \int_0^L e^{-ik_m x} \sin\left(\frac{2\pi x}{L}\right) e^{-ik_n x} dx = \\ &= \frac{V_0}{L} \frac{1}{2i} \int_0^L e^{i(k_m - k_n)x} \left( e^{\frac{i2\pi}{L}x} - e^{-\frac{i2\pi}{L}x} \right) dx = \end{aligned}$$

$$\begin{aligned} &= \frac{V_0}{2iL} \int_0^L e^{i\frac{2\pi}{L}(m-n+1)x} - e^{i\frac{2\pi}{L}(m-n-1)x} dx = \\ &= \frac{V_0}{2iL} [L\delta_{m-n,-1} - L\delta_{m-n,1}] dx = \end{aligned}$$

$$V_{nn} = \frac{V_0}{2iL} \cdot 0 = 0$$

$$V_{n,n+1} = \frac{V_0}{2iL} [L\delta_{1,-1} - L\delta_{1,1}] = \frac{iV_0}{2}$$

$$V_{n,n-1} = \frac{V_0}{2iL} L = -\frac{iV_0}{2}$$

$$V_{nm} = \begin{cases} -\frac{iV_0}{2} & n = m+1 \\ \frac{iV_0}{2} & n = m-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{iV_0}{2} & 0 & \dots \\ -\frac{iV_0}{2} & 0 & \frac{iV_0}{2} & 0 \\ 0 & -\frac{iV_0}{2} & 0 & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix}$$

$$\left[ \underbrace{-\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V(x)} \right] \psi(x) = E' \psi(x)$$

$$(L+V)\psi(x) = E' \psi(x)$$

$$\psi(x) = \sum_{n=1}^2 c_n f_n(x)$$

$$(\mathbf{L} + \mathbf{V})\underline{\mathbf{c}} = E' \underline{\mathbf{c}}$$

$$\mathbf{L}\underline{\mathbf{c}} = E\underline{\mathbf{c}} \quad E_n = \alpha n^2 \quad \alpha \equiv \left( \frac{2\pi^2 \hbar^2}{ML^2} \right)$$

$$\text{i.e.} \quad \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 4\alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & \frac{iV_0}{2} \\ -\frac{iV_0}{2} & 0 \end{pmatrix}$$

$$\left[ \begin{pmatrix} \alpha & 0 \\ 0 & 4\alpha \end{pmatrix} + \begin{pmatrix} 0 & \frac{iV_0}{2} \\ -\frac{iV_0}{2} & 0 \end{pmatrix} - E' \mathbf{I} \right] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$(\alpha - E')(4\alpha - E') - \frac{V_0^2}{4} = 0 \quad V_0 = \sqrt{1.24\alpha}$$

$$(E' - 0.9\alpha)(E' - 4.1\alpha) = 0$$

Eigenvalues:  $E'_{1,2} = 0.9\alpha, 4.1\alpha$

$$E' = 0.9\alpha: \begin{cases} 0.1\alpha c_1 + \frac{iV_0}{2} c_2 = 0 \\ -\frac{iV_0}{2} c_1 + 3.1c_2 = 0 \end{cases}$$

$$c_2 = \frac{i}{\sqrt{31}} c_1 \Rightarrow \underline{\mathbf{c}}_1 = \begin{pmatrix} 1 \\ i/\sqrt{31} \end{pmatrix}$$

$$E' = 4.1\alpha: c_1 = \frac{i}{\sqrt{31}} c_2 \Rightarrow \underline{\mathbf{c}}_2 = \begin{pmatrix} i/\sqrt{31} \\ 1 \end{pmatrix}$$

$$\psi(x) = \sum_{n=1}^2 c_n f_n(x) = c_1 \frac{1}{\sqrt{L}} e^{ik_1 x} + \frac{1}{\sqrt{L}} c_2 e^{ik_2 x}$$

$$\psi_1'(x) = \frac{1}{\sqrt{L}} \left[ e^{ik_1 x} + \frac{i}{\sqrt{31}} e^{ik_2 x} \right]$$

$$\psi_2'(x) = \frac{1}{\sqrt{L}} \left[ \frac{i}{\sqrt{31}} e^{ik_1 x} + e^{ik_2 x} \right]$$

$$\int_0^L \psi_1^*(x) \psi_1(x) dx = 1$$

$$\int_0^L \psi_2^*(x) \psi_2(x) dx = 1$$

$$\text{const} = \frac{1}{\sqrt{c_1^2 + c_2^2}} = \frac{1}{\sqrt{1^2 + \left(\frac{i}{\sqrt{31}}\right)^2}} = \sqrt{\frac{31}{30}}$$

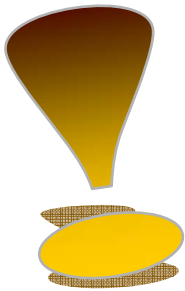
$$\psi_1(x) = \sqrt{\frac{31}{30L}} \left[ e^{ik_1x} + \frac{i}{\sqrt{31}} e^{ik_2x} \right]$$

$$\psi_2(x) = \sqrt{\frac{31}{30L}} \left[ \frac{i}{\sqrt{31}} e^{ik_1x} + e^{ik_2x} \right]$$

## Perturbation Theory

$$\mathbf{A}\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$$

$$L(\underline{r})\Psi_n(\underline{r}) = E_n\Psi_n(\underline{r})$$



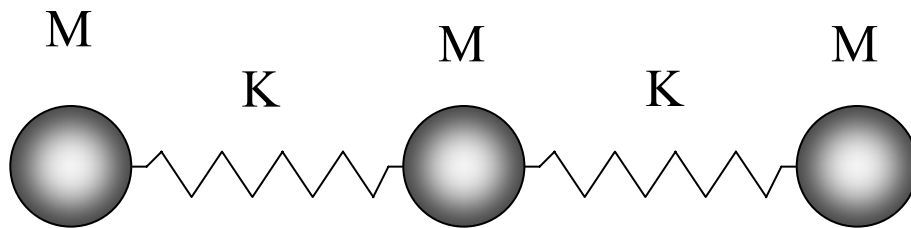
Many problems are not easily solved,  
but are closed to one which can be...

Example:  $L(\underline{r}) = \nabla^2$  (i.e.  $\frac{d^2}{dx^2}$ )

$$\psi_k(\underline{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\underline{r}}$$

$$(\mathbf{A} + \mathbf{a})\underline{\mathbf{x}} = \lambda \underline{\mathbf{x}}$$

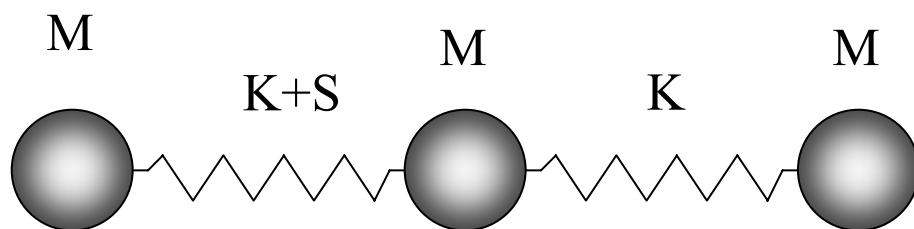
Example:



Described by:  $\mathbf{A}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$

with

$$A = \begin{pmatrix} \frac{K}{M} & -\frac{K}{M} & 0 \\ -\frac{K}{M} & \frac{2K}{M} & -\frac{K}{M} \\ 0 & -\frac{K}{M} & \frac{K}{M} \end{pmatrix}$$



$$\mathbf{B}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$$

$$\mathbf{B} = \begin{pmatrix} \frac{K+S}{M} & -\frac{K+S}{M} & 0 \\ -\frac{K+S}{M} & \frac{2(K+S)}{M} & -\frac{K}{M} \\ 0 & -\frac{K}{M} & \frac{K}{M} \end{pmatrix}$$

$$\mathbf{B} = \mathbf{A} + \mathbf{a}$$

$$\mathbf{a} = \begin{pmatrix} \frac{S}{M} & -\frac{S}{M} & 0 \\ -\frac{S}{M} & \frac{2S}{M} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$$

$$\mathbf{B}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$$

using **Perturbation Theory**

Which says that the solution to  $\mathbf{B}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$   
is a **perturbation** of the solution to  $\mathbf{A}\underline{\mathbf{x}} = \omega^2 \underline{\mathbf{x}}$

$$L(\underline{r})\psi_n(\underline{r}) = E_n\psi_n(\underline{r})$$

$$\int \psi_n^*(\underline{r})\psi_n(\underline{r})d\underline{r} = \delta_{n,m}$$

Need to solve:

$$(L(\underline{r}) + U(\underline{r}))\Psi(\underline{r}) = E\Psi(\underline{r})$$

For  $\Psi(\underline{r})$  and  $E$  given  $U(\underline{r})$ .

Assume that  $U$  is small and  $\Psi(\underline{r})$  is close to an eigenvector  $\psi_n(\underline{r})$ , and  $E$  is close to an eigenvalue  $E_n$ .

i.e.

$$\begin{cases} \Psi = \psi_n + \delta\psi \\ E = E_n + \delta E \end{cases}$$

Expand  $\Psi(\underline{r})$  in terms of the basis:

$$\Psi(\underline{r}) = \sum_m c_m \psi_m(\underline{r}) \Rightarrow$$

$$\Psi(\underline{r}) = \psi_n(\underline{r}) + \sum_{m \neq n} c_m \psi_m(\underline{r}) =$$

$$= \psi_n + \sum_m' c_m \psi_m \quad \left( \sum_m' \text{ means } m \neq n \right)$$

$$\sum_m' c_m \psi_m$$

Substitute for  $\Psi(\underline{r})$  in:

$$(L(\underline{r}) + U(\underline{r}))\Psi(\underline{r}) = E\Psi(\underline{r})$$

$$(L + U)(\psi_n + \sum_m' c_m \psi_m) = E(\psi_n + \sum_m' c_m \psi_m) \Rightarrow$$

$$(L + U - E)(\psi_n + \sum_m' c_m \psi_m) = 0$$

$$\underbrace{(L\psi_n + U\psi_n - E\psi_n)}_{E_n\psi_n} + \sum_m' c_m \underbrace{(L\psi_m - E\psi_m + U\psi_m)}_{E_m\psi_m} = 0$$

$$(E_n - E)\psi_n + U\psi_n + \sum_m' c_m ((E_m - E)\psi_m + U\psi_m) = 0$$

$$\int \psi_m^* \psi_n = \delta_{m,n}$$

$$\begin{aligned} & \overbrace{(E_n - E)}^1 \int \psi_n^* \psi_n + \int \psi_n^* U\psi_n + \\ & + \sum_m' c_m ((E_m - E) \underbrace{\int \psi_n^* \psi_m}_0 + \int \psi_n^* U\psi_m) = 0 \end{aligned}$$

$$E = E_n + \int \psi_n^* U\psi_n + \sum_m' c_m \int \psi_n^* U\psi_m$$

$$0 + \int \psi_p^* U \psi_n + \sum_m' c_m ((E_m - E) \underbrace{\int \psi_p^* \psi_m}_{\delta_{p,m}} + \int \psi_p^* U \psi_m) = 0$$

$$\int \psi_p^* U \psi_n + c_p (E_p - E) + \sum_m' c_m \int \psi_p^* U \psi_m = 0$$

$$c_p = -\frac{\int \psi_p^* U \psi_n}{E_p - E} - \frac{1}{E_p - E} \sum_m' c_m \int \psi_p^* U \psi_m$$

$$E \approx E_n \quad \int \psi_n^* U \psi_m \ll E_n \quad c_m \ll 1$$

$$c_m \int \psi_n^* U \psi_m$$

$$E = E_n + \int \psi_n^* U \psi_n$$

$$c_p = \frac{\int \psi_p^* U \psi_n}{E - E_p}$$

$$\Psi = \psi_n + \sum_m \frac{\int \psi_m^* U \psi_n}{E - E_m} \psi_m$$

$$\Psi = \psi_n + \sum_m \frac{\int \psi_m^* U \psi_n}{E_n - E_m} \psi_m$$

$$E = E_n + \int \psi_n^* U \psi_n +$$

$$+ \sum_m \cdot \frac{\int \psi_m^* U \psi_n \int \psi_n^* U \psi_m}{E - E_m}$$

$$c_m = - \frac{\int \psi_m^* U \psi_n}{E_m - E} -$$

$$- \frac{1}{E_m - E} \sum_p \cdot \frac{\int \psi_m^* U \psi_p \int \psi_p^* U \psi_m}{E - E_p}$$

## The almost harmonic oscillator

$$H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V(x)$$

where

$$V(x) = \frac{1}{2} m \omega^2 x^2 + \epsilon_1 x^3 + \epsilon_2 x^4$$

small

$$(L(x) + U(x))\psi(x) = E\psi(x)$$

Accordingly:

$$L(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2$$

Exact harmonic oscillator

$$\rightarrow U(x) = \varepsilon_1 x^3 + \varepsilon_2 x^4$$

$$E = E_n + E_n^{(1)}$$

first order correction

$$E_n^{(1)} = \int \psi_n^* U(x) \psi_n dx$$

Eigenfunctions corresponding to  $E_n$

In our case:

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega$$

$n = 0$

$$E_0 = \frac{1}{2} \hbar \omega$$

$$\psi_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\begin{aligned}
E_0^{(1)} &= \int_{-\infty}^{+\infty} \psi_0^* U(x) \psi_0 dx = \\
&= \int_{-\infty}^{+\infty} \left( \frac{m\omega}{\hbar\pi} \right)^{\frac{1}{2}} e^{-\frac{m\omega x^2}{\hbar}} (\varepsilon_1 x^3 + \varepsilon_2 x^4) dx = \\
&= \varepsilon_2 \left( \frac{m\omega}{\hbar\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{m\omega x^2}{\hbar}} x^4 dx = \\
&= \frac{3\varepsilon_2 \hbar^2}{4m^2 \omega^2} \quad \Downarrow
\end{aligned}$$

$$E = \frac{1}{2} \hbar \omega + \frac{3\varepsilon_2 \hbar^2}{4m^2 \omega^2}$$