PHY2018 Linear Algebra

Problem class 1

15.45, 15.46, 15.47(a)

Problem class 2

15.63 (c) 15.76 (a, c) 15.82 (a) 15.83 (a) 15.89 (b)

Problem class 3

15.89 (c) 15.96 (a, b, c)

Problem class 4

i) Find eigenvalues and corresponding eigenvectors for matrix

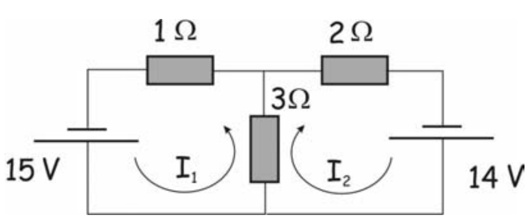
$$A=\begin{pmatrix} 3 & 11 & -11 \\ 1 & 3 & -2 \\ 1 & 5 & -4 \end{pmatrix}$$

ii) Find and perform a transformation which reduces matrix from i) to a diagonal form.

iii) Given the eigenvectors from section i) find unit eigenvectors and check their orthogonality

Problem class 5

1. The combination of currents, resistors and emf are as shown in the picture below. Write a set of linear equations employing currents as independent variables. Find I_1 and I_2 by using matrix operations.



2. Find eigenvalues and eigenvectors of matrix $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

3. Normalize and verify orthogonality of the eigenvectors from section 2. Transform matrix A from section 2 into a canonical form. Which matrices did you use to perform this transformation.

4. **15.87**

Problem class 6

1. Solve the system of coupled differential equations

$$\frac{dy_1}{dt} = -8y_1 + 2y_2$$
$$\frac{dy_2}{dt} = 3y_1 - 3y_2$$

where the y_i (i = 1, 2) are functions of t, subject to the initial conditions $y_1(0) = 1$, $y_2(0) = 10$.

Tips: i) try a solution in form of $\underline{y_i}(t) = \underline{y_i}\exp(\lambda t)$

ii) write in a matrix form $A\underline{y} = \lambda\underline{y}$ where $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

iii) solve the eigenvalue problem, i.e. find the eigenvalues λ_i and corresponding eigenvectors $\underline{y}_i$.

iv) write the most general solution in the form $\underline{Y} = A\underline{y}_1\exp(\lambda_1 t) + B\underline{y}_2\exp(\lambda_2 t)$ and determine the coefficients A and B using the initial conditions.

2. Show that the matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

is both Hermitian and unitary. Find the eigenvalues and eigenvectors of A.

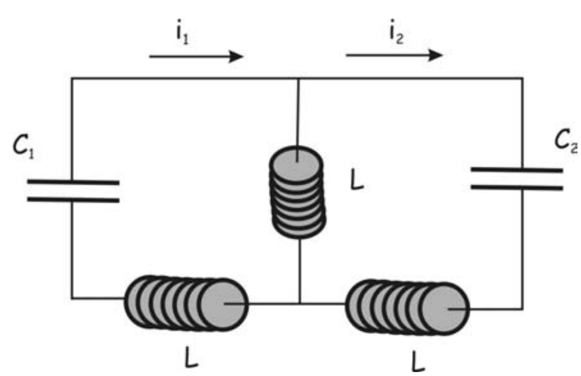
Problem class 7

1. Verify Cayley-Hamilton theorem (15-14), using matrix

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \text{(problem 15.96(b))}$$

Hint: Show that the characteristic equation for matrix A is satisfied when A is used instead of λ (i.e. p(A)=0).

2. The currents in two coupled oscillatory circuits



satisfy the equations

$$L\frac{d^2 i_1}{dt^2}+\frac{i_1}{C_1}+L\left(\frac{d^2 i_1}{dt^2}-\frac{d^2 i_2}{dt^2}\right)=0$$
$$L\frac{d^2 i_2}{dt^2}+\frac{i_2}{C_2}+L\left(\frac{d^2 i_2}{dt^2}-\frac{d^2 i_1}{dt^2}\right)=0$$

a) Assuming periodic solutions, recast these equations into the standard eigenvalue form:

$$A\underline{I}=\lambda\underline{I}$$

Giving espresions for A, $\underline{I}$ and λ.

b) Obtain the eigenvalues and hence the oscillation frequencies.

c) For the case of $C_1 = C_2 = C$

i) obtain the eigenvectors

ii) If at time t=0, $\begin{cases} i_1 = I, i_2 = 0 \\ \frac{di_1}{dt}=0, \frac{di_2}{dt}=0 \end{cases}$ derive an expression for the currents in the two circuits for all subsequent times.

Problem class 8

1. Verify that functions of type $f_n(x) = A\cos(\frac{2\pi n}{L}x)$ where n = 0,1,2 … can be used as basis functions in the interval 0<x<L.

i) Normalise. (find A from the condition $\int_0^L f_n(x)f_n(x)dx = 1$)

ii) Check orthogonality (i.e. show that $\int_0^L f_n(x)f_m(x)dx = \delta_{n,m}$)

iii) Would it be possible to use $f_n(x)$ as the basis for all functions in this interval?

2. Consider the definition of scalar product:

$$(f,g)=\int_{-\infty}^{+\infty} f^*(x)g(x)dx$$

Prove that:

- $(f,\alpha g_1+\beta g_2)=\alpha(f,g_1)+\beta(f,g_2)$
- $(\alpha f_1+\beta f_2,g)=\alpha^*(f_1,g)+\beta^*(f_2,g)$
- $(f,g)=(g,f)^*$

where α and β are complex constants.

3. In Quantum mechanics the notation of "Eigenvalue-eigenvector decomposition" correspond to an opposite of "Diagonalisation". If eigenvalues and eigenvectors are known the result of the decomposition is a matrix representing the **Observable** of the quantity in question.

e.g. for the following decomposition: $A = B\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} B^{-1}$ where $\lambda_{1,2,3}$ are the eigenvalues and B is the matrix of eigenvectors, matrix A will be the observable.

Find the observable of the angular momentum of a circularly polarised photon knowing that it can have only two possible states:

a) right-hand polarisation - with the normalised eigenvector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and the corresponding eigenvalue $\hbar$

b) left-hand polarisation - with the normalised eigenvector $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and the corresponding eigenvalue $-\hbar$.

(Hint: might help to recall that for unitary matrices $\overline{B}^T = B^{-1}$)

Problem class 9

1. Consider a differential equation of the following form:

$$\frac{d^2\Psi(x)}{dx^2} + \lambda\Psi(x) = 0 \text{ with the boundary conditions } \Psi(0)=\Psi(\pi)=0$$

Find possible values of λ by approaching this equation as an eigenvalue problem.

i) Choose an appropriate set of basis functions. (use sin() series, but decide what is going to be in the argument (the boundary conditions might help!!!))
ii) Normalise the basis functions and verify orthogonality
iii) With respect to the basis (i.e. having expanded function $\Psi(x)$ by basis functions) write the differential equation as a matrix eigenvalue problem.
iv) Find the eigenvalues.
v) Show that the eigenvalues agree with the possible values of λ after direct substitution of $\Psi(x)$ into the differential equation.

Problem class 10

1. The energy eigenfunctions and eigenvalues of a particle of mass m confined to the region $0 \le x \le L$ by infinite potential barriers are

$$\Psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \quad \text{and} \quad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

Calculate the first-order correction to the energy of the 1^{st} level (i.e. to the eigenvalue E_1) in the case of a small perturbation U(x) to the potential:

Where: U(x) = $\begin{cases} C\sin\left(\frac{2\pi x}{L}\right) & 0 \le x \le L \\ 0 & otherwise \end{cases}$

Problem class 11

1. Find an inverse matrix for:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

2. Find characteristic equation and eigenvalues for matrix A:

$$A = \begin{pmatrix} 3 & 6 & 2 \\ 0 & -3 & -8 \\ 1 & 0 & -4 \end{pmatrix}$$

3. For the given system of springs and masses

Where $M_1 = \frac{1}{2}$ M, $M_2 = M$, $M_3 = \frac{1}{2}$ M, $M_4 = M$
$K_1=2K$, $K_2 = K$, $K_3 = 2K$, $K_4 = K$.

a) write a system of differential equations corresponding to displacements of masses x1, x2, x3, x4.
b) Assuming periodic solutions $x_k = x_k e^{i\omega t}$ (k = 1, 2, 3, 4), rewrite the system in the standard eigenvalue form:

$$A\underline{x} = \lambda\underline{x}$$