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Errors:

What they are, and how to deal with them

Two 2-hour sessions in support of
first-year laboratory

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Lab experiments in a nutshell

Make some measurements → What is the error in these measurements?

Calculate some useful physical quantity from them → How do these ‘propagate’ to an error in the calculated quantity

Compare with ‘expected’ value → Is your answer within errors of the expected one?

(Might the ‘expected’ value also have an error?)

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Synopsis

1) Introduction:
   - what is an error?
   - estimating the error in a measurement

2) Rules for quoting errors

3) Combining errors

4) Statistical analysis of random errors

5) Use of graphs in experimental physics

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Aims of this lecture

- What is an error
- two types of errors – random and systematic
- estimating measurement errors
- quoting errors

1) Introduction

What is an error?

All measurements in science have an uncertainty (or "error") associated with them:

- Errors come from the limitations of measuring equipment, or
- in extreme cases, intrinsic uncertainties built in to quantum mechanics

Errors can be minimised, by choosing an appropriate method of measurement, but they *cannot be eliminated*.

Notice that errors are not "mistakes" or "blunders".

Two types of error:

Random and **Systematic**

Random: Errors which cause the measurement to be as often larger than the true value, as it is smaller. Taking the average of several measurements of the same quantity reduces this type of error (as we shall see later).

Example: the reading error in measuring a length with a metre rule.

Systematic: Errors which cause the measurement always to differ from the truth in the same way (i.e. the measurement is always larger, or always smaller, than the truth).

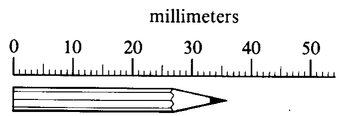
Example: the error caused by using a poorly calibrated instrument.

Systematic errors are often the reason why the result you calculate from your measurement is not consistent with the expected value

It is random errors which we will be considering in this discussion, because they are susceptible to mathematical treatment

Estimating errors

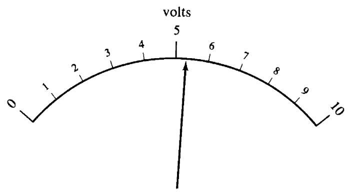
e.g. Measurements involving marked scales:



Closer to 36 mm than to 35 or 37, so we state:

best estimate of length	=	36 mm
probable range		35.5 to 36.5 mm

Estimating errors (2)



The markings on the voltmeter are more widely spaced than on the ruler.

State the reading to a better accuracy than to the nearest mark:

best estimate of voltage	=	5.3 volts
probable range		5.2 to 5.4 volts

The process of estimating positions between markings is called "interpolation"

Estimating errors (3)

Repeatable measurements

There is a sure-fire way of estimating errors when a measurement can be repeated several times. e.g. measuring the period of a pendulum, with a stopwatch.

e.g. Four measurements are taken of the period of a pendulum. They are:

2.3 sec, 2.4 sec, 2.5 sec, 2.4 sec

In this case:

best estimate = average	=	2.4 sec
probable range		2.3 to 2.5 volts

Repeating measurements has other benefits – see later

2) Rules for quoting errors

The result of a measurement of the quantity x should be quoted as:

$$(\text{measured value of x}) = x_{best} \pm \delta x$$

where x_{best} is the best estimate of x, and the probable range is from $x_{best} - \delta x$ to $x_{best} + \delta x$.

We will be more specific about what is meant by "probable range" later.

Significant figures

Examples of BAD practice

measured g = $9.82 \pm 0.02385\ ms^{-2}$

error contains too many significant figures

measured speed = $6051.78 \pm 30\ ms^{-1}$

best estimate contains too many significant figures

Significant figures (2)

The rules

- Experimental uncertainties should normally be rounded to one significant figure.
- The last significant figure of the answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

For example the answer 92.81 with error of 0.3 should be stated:

92.8 ± 0.3

If the error were 3, then the same answer should be stated as:

93 ± 3

If the error were 30 then state as: **90 ± 30**

Significant figures (3)

One exception

If the leading digit of the error is small (i.e. 1 or 2) then it is acceptable to retain one extra figure in the best estimate.

e.g. “measured length = 27.6 ± 1 cm” is as acceptable as “28 ± 1 cm”

Quoting measured values involving exponents

It is conventional (and sensible) to do as in the following example:

suppose the best estimate of the measured mass of an electron is 9.11×10^{-31} kg with an error of $\pm 5 \times 10^{-33}$ kg, then the result should be written:

measured electron mass = $9.11\ (\pm 0.05) \times 10^{-31}$ kg

(i.e. NOT $9.11 \times 10^{-31}\ (\pm 5 \times 10^{-33})$ kg)

This makes it easy to see how large the error is compared with the best estimate.

Aside: a convention

In science, the statement “x = 1.27” (without any statement of error) is taken to mean $1.265 \leq x \leq 1.275$

To avoid any confusion, best practice is always to state an error, with every measurement.

Absolute and fractional errors

The errors we have quoted so far have been absolute ones (they have had the same units as the quantity itself).

It is often helpful to quote errors as a fraction (or percentage) of the best estimate. In other words, in the form:

$$\text{fractional error in x} = \frac{\delta x}{|x_{best}|}$$

(the modulus simply ensures that the error is positive)

> The fractional error is a measure of the quality of a result.

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Absolute and fractional errors (2)

In terms of fractional errors one would then quote a measurement as:

$$\text{measured value of x} = x_{best}\left(1 \pm \frac{\delta x}{|x_{best}|}\right)$$

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Comparison of measured and accepted values

It is often the objective of an experiment to measure some quantity and then compare it with the accepted value (e.g. a measurement of g, or of the speed of light).

e.g. in an experiment to measure the speed of sound in air, we come up with the result:

Measured speed = $329 \pm 5\ ms^{-1}$

The accepted value is $331.0\ ms^{-1}$.

CONCLUSION: measured speed is CONSISTENT with the accepted value, because the value 331 lies within the range 329-5 and 329+5.

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Comparison of values (2)

What if the measured speed was $345 \pm 5\ ms^{-1}$?

INCONSISTENT

What if the measured speed was $329 \pm 1\ ms^{-1}$?

INCONSISTENT

(even though value is the same as in first example – errors matter!)

MUST look for reasons for any inconsistency:

e.g. a systematic error in the experiment

or an actual mistake has been made

Note: accepted values can also have errors

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Comparison of values (3)

If the error quoted is a standard deviation (more on what this is later), then there is a ~68% chance that a single measurement will be within one standard deviation of the accepted value.

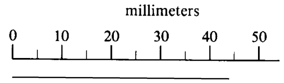
⇒ 32% chance that it lies outside this range

possible reason for (small) 'inconsistency'

Note that there is a 95% chance of a single measurement being within TWO standard deviations

Quiz on last lecture

- What is the length of this line?



44 ± 1 mm, or 44.0 ± 0.5 mm

- In a measurement of g, the value obtained was 9.81 ms^{-2} with a ±4% error. How would you write this result?

9.8 ± 0.4 ms^{-2}

- In a measurement of the circumference of the earth, the value obtained was 40,000 km, with an error of 700 km. How would you write this, using exponential notation?

4.00 (±0.07) $\times 10^4$ km

Error in a sum or difference

The desired quantity q is related to two measured quantities x and y by

$$q = x + y$$

The measured value of x is $x_{best} \pm \delta x$, and that of y is $y_{best} \pm \delta y$.

What is the best estimate q_{best}, and the error δq, in q?

The highest probable value of x + y is: $x_{best} + y_{best} + (\delta x + \delta y)$

The lowest probable value of x + y is: $x_{best} + y_{best} - (\delta x + \delta y)$

so $q_{best} = x_{best} + y_{best}$ and $\delta q = \delta x + \delta y$

Error in a sum or difference (2)

Check for yourselves that a similar argument shows that the error in the difference x - y is also dx + dy.

Error in a product or quotient

This time the quantity q is related to two measured quantities x and y by:

$$q = x \times y$$

Writing x and y in terms of their fractional errors:

measured value of x = $x_{best}\left(1 \pm \frac{\Delta x}{|x_{best}|}\right)$ and of y = $y_{best}\left(1 \pm \frac{\Delta y}{|y_{best}|}\right)$

so the largest probable value of q is $x_{best}y_{best}\left(1 + \frac{\Delta x}{|x_{best}|}\right)\left(1 + \frac{\Delta y}{|y_{best}|}\right)$

while the smallest probable value is $x_{best}y_{best}\left(1 - \frac{\Delta x}{|x_{best}|}\right)\left(1 - \frac{\Delta y}{|y_{best}|}\right)$

Error in a product or quotient (2)

Just consider the two bracketed terms in the first of these:

$$\left(1 + \frac{\Delta x}{|x_{best}|}\right)\left(1 + \frac{\Delta y}{|y_{best}|}\right) = 1 + \frac{\Delta x}{|x_{best}|} + \frac{\Delta y}{|y_{best}|} + \frac{\Delta x}{|x_{best}|}\frac{\Delta y}{|y_{best}|}$$

The last term contains the product of two small numbers, and can be neglected in comparison with the other three terms.

so largest probable value of q is $x_{best}y_{best}\left(1 + \frac{\Delta x}{|x_{best}|} + \frac{\Delta y}{|y_{best}|}\right)$

similarly, the smallest probable value is $x_{best}y_{best}\left(1 - \left(\frac{\Delta x}{|x_{best}|} + \frac{\Delta y}{|y_{best}|}\right)\right)$

Error in a product or quotient (2)

Hence $q_{best} = x_{best} \times y_{best}$ and $\frac{\Delta q}{|q_{best}|} = \frac{\Delta x}{|x_{best}|} + \frac{\Delta y}{|y_{best}|}$

For the case of $q = \frac{x}{y}$ exactly the same result for the error in q is obtained.

Summary

For sums and differences, the <u>absolute</u> error in the result is obtained by adding the <u>absolute</u> errors in the original quantities.

remember the absolute error in x is Δx

it has the same units as x

For products and quotients, the <u>fractional</u> error in the result is obtained by adding the <u>fractional</u> errors in the original quantities.

remember the fractional error in x is $\frac{\Delta x}{x}$

it has no units (dimensionless)

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The error in a power

Suppose $\quad q = x^n$

Now x^n is just a product of x with itself (n times).
So applying the rule for errors in a product, we obtain:

$$\frac{\delta q}{|q_{best}|} = n\frac{\delta x}{|x_{best}|}$$

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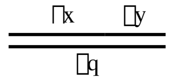
Refinement of the rules for independent errors

Actually we have been a bit pessimistic by saying that we should add the errors:

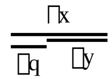
If x and y are independent quantities, the error in x is just as likely to (partially) cancel the error in y, as it is to add to it.

add the errors:



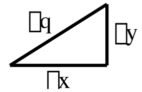
Pessimistic

subtract the errors:



Optimistic

add the errors in quadrature:



Correct

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Refinement of the rules, independent errors (2)

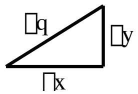
In other words, instead of saying (for the case of sums and differences):

$$\delta q = \delta x + \delta y$$

we say:

$$\delta q = \sqrt{\delta x^2 + \delta y^2}$$

By Pythagoras: δq is the hypotenuse of a right angled triangle



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Refinement of the rules, independent errors (3)

Comments about adding in quadrature: $\delta q = \sqrt{\delta x^2 + \delta y^2}$

- the error in the result, δq is greater than the errors in either of the measurements, but
- it is always less than the sum of the errors in the measurements.
- the process of adding in quadrature has reduced the error (compared with simple addition), as we required.
- It is possible to show rigorously that this is the correct procedure for combining errors.

Summary of refined rules

For sums and differences, the absolute error in the result is obtained by taking the root of the sum of the squares of the absolute errors in the original quantities.

For products and quotients, the fractional error in the result is obtained taking the root of the sum of the squares of the fractional errors in the original quantities.

NB: the rule for powers remains as before, because each of the ‘x’s in x^n are the same (and have the same error), so this is not a product of independent quantities.

Other important cases

$q = Bx$: Product of a measured quantity with an exact number

Treat as a product but B has no error, so $\frac{\delta q}{q} = \frac{\delta x}{x}$ or $\boxed{\delta q = B\delta x}$

Combinations of sums, differences and powers, e.g. $q = x\left(y + z^2\right)$

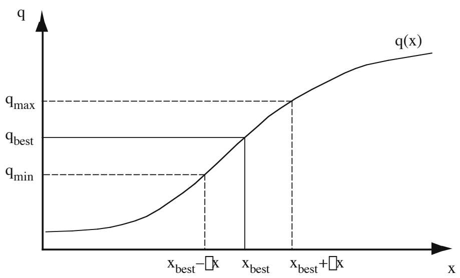
Split the function up and work out the error in its separate components:

- first calculate the error in z^2 – use the rule for powers
- then use this to find the error in $y + z^2$ – rule for sums
- finally combine this with the error in x – rule for products

BUT see also the following.....

The error in any function of one variable

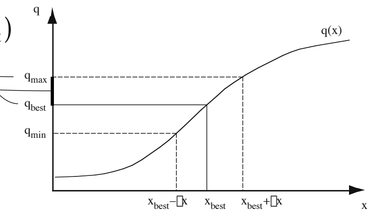
Here the quantity we want to calculate, q, is related to a quantity we measure, x, by: $q = q(x)$ (i.e. q is any function of x)



dashed lines show how the error in the measured quantity x affects the error in q

The error in any function... (2)

Now $\delta q = q\left(x_{best} + \delta x\right) - q\left(x_{best}\right)$



But from calculus, we know that the definition of $\frac{dq}{dx}$ is:

$$\frac{dq}{dx} = \text{limit as } \delta x \to 0 \left[\frac{q(x + \delta x) - q(x)}{\delta x}\right]$$

The error in any function... (3)

For a sufficiently small error in the measured quantity x, we can say that the error in any function of one variable is given by:

$$\delta q = \left|\frac{dq}{dx}\right| \delta x$$

(where the modulus sign ensures that δq is positive - by convention all errors are quoted as positive numbers)

But from calculus, we know that the definition of $\frac{dq}{dx}$ is:

$$\frac{dq}{dx} = \text{limit as } \delta x \to 0 \left[\frac{q(x+\delta x) - q(x)}{\delta x}\right]$$

General formula for combining uncertainties

We can extend this to calculating the error for a function of more than one variable, by realising that if $q = q(a,...,z)$ (several measured variables) then the effect that an error in one measured quantity (say the variable u) is given by:

$$\delta q(\text{u only}) = \left|\frac{\partial q}{\partial u}\right| \delta u$$

Notation: $\frac{\partial q}{\partial u}$ is called the partial derivative of the function q with respect to u.

It is calculated by differentiating q wrt u, treating all the other variables as if they were constants.

General formula for combining uncertainties

So the total error is obtained by combining these using addition in quadrature (assuming the errors are all independent):

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial a}\delta a\right)^2 + ... + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

- Evaluate the error caused by each of the individual measurements, and then
- combine them by taking the root of the sum of the squares.

The rules for the special cases can be derived from this formula

Summary of this lecture

Rules for combining errors:

- For sums and differences: $\delta q = \sqrt{\delta x^2 + \delta y^2}$
- For products and quotients: $\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$
- For powers: $\frac{\delta q}{q} = n\frac{\delta x}{x}$
- For a general function: $\delta q = \sqrt{\left(\frac{\partial q}{\partial a}\delta a\right)^2 + ... + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$

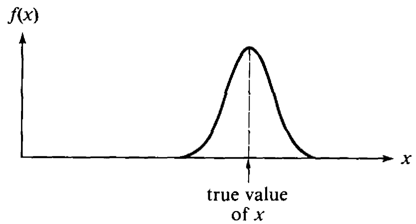
Aims of this lecture

- Statistical analysis of random errors:
  - » mean
  - » standard deviation in a measurement
  - » normal distribution
  - » standard deviation in the mean
- Plotting graphs:
  - » choosing what to plot
  - » plotting graphs by hand
  - » the least-squares fitting method

4) Statistical analysis of random errors

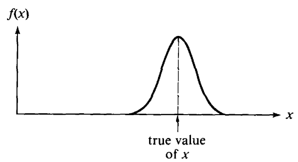
In this section, we discuss a two of useful quantities in dealing with random errors, the mean and the standard deviation.

It is helpful first to consider the distribution of values obtained when a measurement is made:



Typically this will be 'bell-shaped' with a peak near the average value.

Statistics of errors (2)



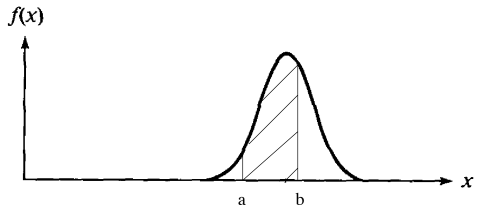
For measurements subject to many sources of small random errors (and no systematic error) this distribution is called a Gaussian, or Normal Distribution and has the mathematical form:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-X)^2}{2\sigma^2}\right]$$

- The peak is at x = X
- The width of the bell-shape is proportional to σ

Statistics of errors (3)

The point of a distribution function is that the probability of a measurement giving a value between two limits is proportional to the are under the curve between the limits:



Area shown gives probability of a measurement yielding a value between a and b.

Statistics of errors (4)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-X)^2}{2\sigma^2}\right]$$

Further comments about the Normal Distribution

- The $\frac{1}{\sigma\sqrt{2\pi}}$ in front of the exponential is called the normalisation constant. It makes areas under the curve equal to the probability (rather than just proportional to it).
- The shape of the Gaussian Distribution indicates that a measurement is more likely to yield a result close to the mean than far from it.
- The rule for adding independent errors in quadrature is a property of the Normal Distribution.
- The concepts of the mean and standard deviation have useful interpretations in terms of this distribution.

The mean

We have already agreed that a reasonable best estimate of a measured quantity is given by the average, or mean, of several measurements:

$$x_{best} = \bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N}$$

$$= \frac{\sum x_i}{N}$$

where the Σ ("sigma") simply denotes the sum of all the individual measurements, from x_1 to x_N.

The mean of the Normal Distribution coincides with the peak, x=X. (Not true of all distributions)

The standard deviation

This takes the role of an 'average' error of a single measurement.

The deviation of the measurement x_i from the mean is $d_i = x_i - \bar{x}$.

If we simply average the deviations, we will get zero – the deviations are as often positive as negative.

A more useful indicator of the average error is to square all the deviations before averaging them, but then to take the square root of the average (so that the resulting error has the correct units)

Standard deviation (2)

This is termed the "ROOT MEAN SQUARE DEVIATION", or the "STANDARD DEVIATION":

$$\sigma_N = \sqrt{\frac{\sum (d_i)^2}{N}}$$

$$= \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$$

Minor complication: you may come across another type of standard deviation defined as: $\sigma_{N-1} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N-1}}$

This turns out to be better especially for small N – don't worry about it!

Standard deviation (3)

If the measurement follows a Normal Distribution, the the standard deviation is just the σ in

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - X)^2}{2\sigma^2}\right]$$

(We already saw that the width of the bell-shaped curve is governed by σ)

If we calculate the area under the curve between $x-\sigma$ and $x+\sigma$ we find that

there is a 68% probability of a single measurement giving a result within one standard deviation of the mean.

Standard deviation in the mean

So far we have discussed the standard deviation of a single measurement.

If we take the mean of N measurements, the standard deviation of the mean is $\sqrt{N}$ times LESS than the standard deviation in a single measurement:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

To reduce random errors take the mean of many measurements

Again one can prove this to be a property of Normal Distributions

5) Use of graphs in experimental physics

Two reasons

The first is to test the validity of a theoretical relationship between two quantities.

Example

Simple theory says that if a stone is dropped from a height h and takes a time t to fall to the ground, then:

$$h \propto t^2$$

If we took several different pairs of measurements of h and t, and plotted h vs. t^2 the theory would be confirmed if the result was a straight line.

Graphs (2)

The second reason for plotting graphs is to determine a physical quantity from a set of measurements of two variables having a range of values.

Example

In the example of the time of flight of a falling stone, the theoretical relationship is, in fact:

$$h = \frac{1}{2} g t^2$$

If we assume this to be true we can measure t for various values of h, and then a plot of h vs. t^2 will have a gradient of $\frac{g}{2}$.

Graphs (3) – choosing what to plot

In general the aim is to make the quantity which you want to know the gradient of a straight-line graph.

In other words obtain an equation of the form:

$$y = mx + c$$

with m being the quantity of interest.

In the example above we plotted h vs. t^2 because this gave us the quantity we wanted, g, in the gradient (g = 2m).

Graphs (4) – choosing what to plot: examples

What should one plot in order to determine the constant □ from the measured quantities x and y, related by the expression:

y = Aexp(□x) Answer: ln(y) vs. x

What if one wanted the quantity A, and □ is already known?

Answer: y vs. exp(□x)

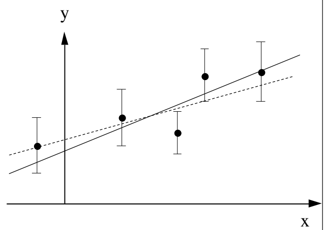
What if one wanted to know the exponent n in the expression:

$y = x^n$ Answer: ln(y) vs. ln(x)

Graphs (5) – plotting by hand

Although you will often use a computer to plot straight line graphs for you, it is perfectly straightforward to determine the best-fit gradient and intercept, and the error in these quantities, on a graph plotted by hand.

Graphs (6) – plotting by hand...



- Plot the points with their error bars
- to draw the best line, position your ruler so that there is a random scatter of points above and below the line

  – about 68% of the points should be within error-bars of the line
- draw the "worst-fit" line: Swivel your ruler about the "centre of gravity" of your best-fit line until the deviation of points (including their error bars) becomes unacceptable
- error in the gradient is □m = $|m_{best\ fit}$ □ $m_{worst\ fit}|$

  (similar expression for intercept)

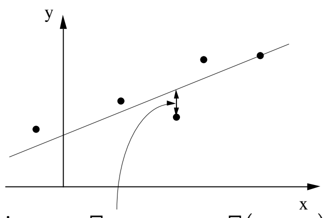
Graphs (7) – the least-squares fit

This is the mathematical procedure for finding m and c for the equation:

$$y = mx + c$$

of the straight line which most closely fits a given set of data $[x_n, y_n]$.

Specifically, the procedure minimises the sum of the squares of the deviations of each value of y_n from the line:



$$\text{deviation} = y_n - y_{\text{best fit}} = y_n - (mx + c)$$

Graphs (8) – the least-squares fit...

The fitting procedure obtains values of m and c, and also standard deviations in m and c, σ_m and σ_c.

Points to note:

- The procedure assumes that the x measurements are accurate, and that all the deviation is in y. Bear this in mind when deciding which quantity to plot along which axis.

  If the two measured quantities are subject to similar errors then it is advisable to make two least-squares fits, swapping the x and y axes around. Then

$$\sigma_m(\text{total}) = \sqrt{\sigma_m(\text{x vs. y})^2 + \sigma_m(\text{y vs. x})^2}$$

Graphs (9) – the least-squares fit...

Points to note...

- The least-squares fit assumes that the errors in all the individual values of y_n are the same, and is reflected in the scatter of the points about the line. It does not allow you to put different error bars on each point.

  You can plot graphs with error bars on the computer, but the fitting procedure ignores them.

- FOR THE ENTHUSIAST: the fitting procedure comes up with analytical expressions for m, c, σ_m and σ_c, (i.e. it isn't a sort of trial and error procedure which only the computer can do). In principle one could calculate these by hand.

Summary of this lecture

- Definitions of:

the mean $\bar{x} = \frac{\sum x_i}{N}$

the standard deviation $\sigma_N = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$

the standard deviation in the mean $\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$

...and their relation to the Normal Distribution

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Summary of this lecture...

- Plotting graphs:

  » choosing what to plot – make the desired quantity the gradient of a straight-line graph

  » plotting graphs by hand – use the best and worst fit line to find errors

  » the least-squares fitting method – mathematical method of finding gradients, intercepts and their standard deviations
  BUT be aware of its limitations

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