Transmission Lines

Introduction

A *transmission line* guides energy from one place to another. Optical fibres, waveguides, telephone lines and power cables are all electromagnetic transmission lines. Transmission lines are usually analysed by approximating them by a chain of many 2-port devices (figure 1). The 2-port elements can be represented by lumped circuits such as those in figure 2. We will start by looking at the lossless line in terms of the equivalent-T lumped circuit model (figure 3). The quantities L and C are respectively the inductance and capacitance *per unit length* of the line. Bold symbols in this document indicate complex quantities.

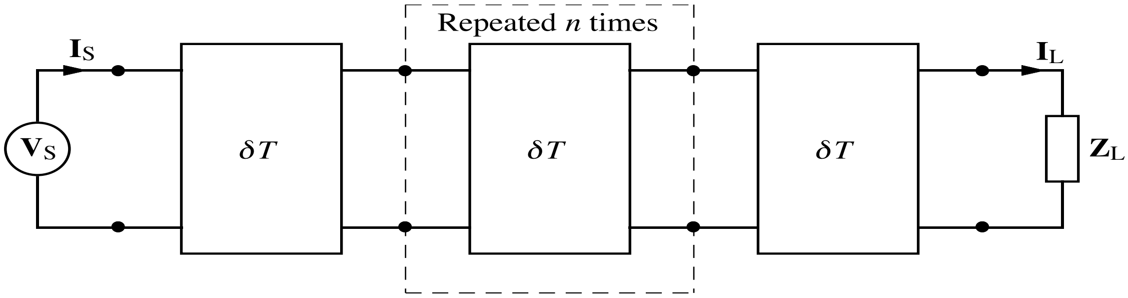

Figure 1. A transmission line system approximated by a chain of 2-port networks.

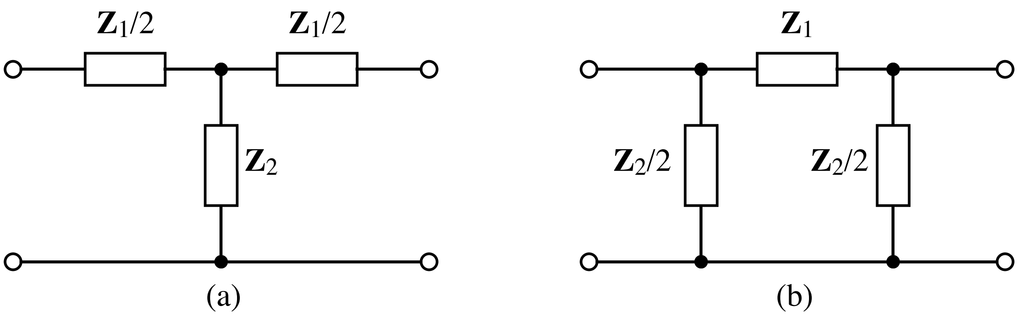

Figure 2. Two common lumped impedance representations for the 2-port networks

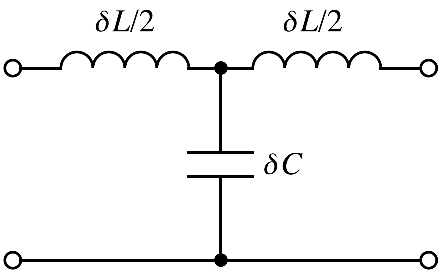

Figure 3. A 2-port network representing a length δ of lossless line.

Characteristic Impedance

If an infinite line has characteristic impedance $\mathbf{Z}_0 = \mathbf{V}_S / \mathbf{I}_S$ then adding one extra section will not alter this value. The impedance between terminals A and B of the circuit shown in figure 4 must therefore also be $\mathbf{Z}_0$ and can be found by solving

$$\mathbf{Z}_0 = \tfrac{1}{2}\mathbf{Z}_1 + \mathbf{Z}_2 \| \left(\tfrac{1}{2}\mathbf{Z}_1 + \mathbf{Z}_0\right) \quad \Rightarrow \quad \mathbf{Z}_0^2 = \mathbf{Z}_1\mathbf{Z}_2 + \tfrac{1}{4}\mathbf{Z}_1^2 . \tag{1}$$

In the idealised case of a lossless line $\mathbf{Z}_1 = j\omega\delta L$ and $\mathbf{Z}_2 = 1/j\omega\delta C$

$$\mathbf{Z}_0 = \lim_{\delta \to 0} \sqrt{(L/C) - (\delta\,\omega L/2)^2} = \sqrt{(L/C)} . \tag{2}$$

A generator with a real source impedance R_S will transmit maximum power to a load having an impedance equal to R_S *via* a lossless transmission line of characteristic impedance $\mathbf{Z}_0 = R_S$ independent of frequency.

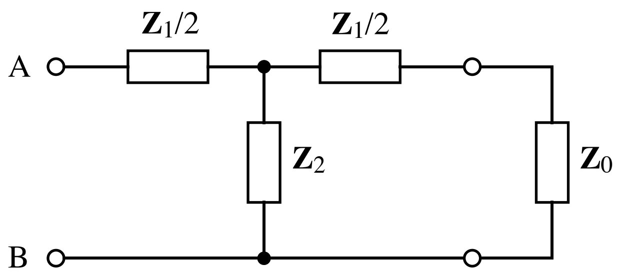

Figure 4. An extra section is added to an infinite line.

Coaxial Cables

Consider a coaxial cable comprising an inner conductor of radius a, enclosed in a conducting cylinder of inner radius b with the gap filled with dielectric material of relative permittivity ε_r. The capacitance per unit length of this cable is

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)} . \tag{3}$$

The inductance per unit length L, can be found by equating to $\tfrac{1}{2}LI^2$ the energy stored in the magnetic field created by a current I flowing along the inner conductor and back along the shield. The magnetic field is

$$B_r = \frac{\mu_0 I}{2\pi r} \quad \text{for} \quad a < r < b \tag{4}$$

so

$$\tfrac{1}{2}LI^2 = \frac{1}{\mu_0}\int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 2\pi r\,\mathrm{d}r = \frac{\mu_0 I^2}{4\pi}\ln(b/a) \quad \Rightarrow \quad L = \frac{\mu_0}{2\pi}\ln(b/a) \tag{5}$$

and therefore the characteristic impedance of such cable is

$$\mathbf{Z}_0 = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}}\,. \tag{6}$$

Example: Type RG-58 cable which is commonly used in laboratories has polythene dielectric with $\varepsilon_r = 2.3$ and a=0.5 mm, b=1.75 mm from which we can estimate that: C=95 pF m^{-1}, L=270 nH m^{-1}, $\mathbf{Z}_0$=53 Ω, and a phase velocity $v=2\times10^8$ m s^{-1}.

Wave Propagation

If the voltage at the centre of the element at position x is $V(x)$ then the centre of the next element is at position $x+\delta$ and

$$V(x+\delta) - V(x) = -L\delta\,\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7}$$

so in the limit as $\delta \to 0$

$$\left(\frac{\partial V}{\partial x}\right) = -L\left(\frac{\partial I}{\partial t}\right) \quad \text{and similarly} \quad \left(\frac{\partial I}{\partial x}\right) = -C\left(\frac{\partial V}{\partial t}\right) \tag{8a,b}$$

differentiating one of these with respect to x and the other with respect to t and combining the results in two different ways gives

$$\left(\frac{\partial^2 V}{\partial x^2}\right) = LC\left(\frac{\partial^2 V}{\partial t^2}\right) \quad \text{and similarly} \quad \left(\frac{\partial^2 I}{\partial x^2}\right) = LC\left(\frac{\partial^2 I}{\partial t^2}\right). \tag{9a,b}$$

These are wave equations and have solutions that are superpositions of waves travelling away from and towards the source

$$V(x,t) = V_+(x - vt) + V_-(x + vt) \quad \text{and} \quad I(x,t) = I_+(x - vt) + I_-(x + vt) \tag{10a,b}$$

where V_+, V_-, I_+, I_- are arbitrary functions and the phase velocity of propagation v is

$$v = \frac{1}{\sqrt{LC}}\,. \tag{11}$$

The signal source and load will impose boundary conditions that place some constraints on the relationships between the ‘arbitrary’ functions.

Unmatched Loads

The wave equations 9a,b are linear so if the source generates a signal at a single frequency the solutions must be of the form

$$\mathbf{V}(x,t) = \mathbf{A}\exp\big(j(\omega t - kx)\big) + \mathbf{B}\exp\big(j(\omega t + kx)\big) \tag{12}$$

and since equations 8a,b must be satisfied

$$\mathbf{I}(x,t) = \frac{1}{\mathbf{Z}_0}\Big[\mathbf{A}\exp\big(j(\omega t - kx)\big) - \mathbf{B}\exp\big(j(\omega t + kx)\big)\Big]. \tag{13}$$

If at x=0 there is a terminating impedance $\mathbf{Z}_\mathrm{L}$ then the voltage and current at this point must satisfy

$$\mathbf{V}(0,t) = \mathbf{I}(0,t)\mathbf{Z}_L \tag{14}$$

$$\Rightarrow \quad \mathbf{Z}_0 \exp(j\omega t)[\mathbf{A} + \mathbf{B}] = \mathbf{Z}_\mathrm{L} \exp(j\omega t)[\mathbf{A} - \mathbf{B}] \tag{15}$$

$$\Rightarrow \quad \mathbf{K} = \frac{\mathbf{B}}{\mathbf{A}} = \frac{\mathbf{Z}_\mathrm{L} - \mathbf{Z}_0}{\mathbf{Z}_\mathrm{L} + \mathbf{Z}_0} \tag{16}$$

which defines the *reflection coefficient* **K** as the ratio of signal incident on the load to signal reflected back towards the source. When $\mathbf{K} \neq 0$ the terminating impedance said to be an *unmatched load* and there are standing waves set up along the transmission line and it is easy to show that the ratio of maximum to minimum voltage amplitude along the line, the *voltage standing wave ratio* (VSWR), is

$$\mathrm{VSWR} = \frac{1 + |\mathbf{K}|}{1 - |\mathbf{K}|}\text{`} \tag{17}$$

Total reflection occurs when the load absorbs no power, which can happen when the load is a short- $(\mathbf{K} = -1)$ or open- $(\mathbf{K} = 1)$ circuit, or is purely reactive $(|\mathbf{K}| = 1)$. These cases can be distinguished experimentally because the positions of the minima vary with respect to the load.

The phase of the reflected wave is independent of frequency so if the signal source generates a pulse the reflected signal will be a pulse of identical shape, but generally different amplitude, to the original.

Input impedance of an unmatched line

The input impedance of a line of length l with a load $\mathbf{Z}_\mathrm{L}$ at x=0 is

$$\mathbf{Z}_\mathrm{IN} = \frac{\mathbf{V}(-l,t)}{\mathbf{I}(-l,t)} = \frac{\mathbf{A}\exp(j\omega t)\big[\exp(jkl) + \mathbf{K}\exp(-jkl)\big]}{(1/\mathbf{Z}_0)\mathbf{A}\exp(j\omega t)\big[\exp(jkl) - \mathbf{K}\exp(-jkl)\big]} \tag{18}$$

and using equation 16 to substitute for **K**

$$\mathbf{Z}_\mathrm{IN} = \mathbf{Z}_0\left(\frac{\mathbf{Z}_\mathrm{L}\cos kl + j\mathbf{Z}_0 \sin kl}{\mathbf{Z}_0 \cos kl + j\mathbf{Z}_\mathrm{L}\sin kl}\right). \tag{19}$$

There are three interesting special cases: (1) When the load is a short-circuit

$$\mathbf{Z}_{\mathrm{IN}} = j\mathbf{Z}_0 \tan kl\ . \tag{20}$$

and since a lossless line has a real $\mathbf{Z}_0$ the input impedance is purely reactive. (2) When the load is an open-circuit

$$\mathbf{Z}_{\mathrm{IN}} = -j\mathbf{Z}_0 \cot(kl) = j\mathbf{Z}_0 \tan(kl - \pi/2). \tag{21}$$

which is the same as for a short-circuit at a position shifted by a quarter of a wavelength. (3) When the line is a quarter of a wavelength long, *i.e.* $kl = \pi/2$ and

$$\mathbf{Z}_{\mathrm{IN}} = \frac{\mathbf{Z}_0^2}{\mathbf{Z}_{\mathrm{L}}} \tag{22}$$

so the input impedance is real when a quarter-wavelength line is terminated by *any* real impedance. This has the important application that two transmission lines of different characteristic impedance can be joined without causing reflections if a suitable *quarter-wave transformer* is used between the two lines. This technique obviously only works for a fairly narrow band of frequencies specific to a given transformer and can also be used for matching inconvenient loads.

Practical Transmission Lines

The preceding discussion implicitly considered a transmission line with one propagation mode. Real transmission lines normally exhibit *multimode propagation*. For example, a twisted pair transmission line has two eigenmodes, often referred to as the *differential* and *common* modes. These propagate with different velocities and characteristic impedances. In practical applications, in order to maintain *signal integrity*, it is important that the line-drivers launch only the appropriate single eigenmode in order to avoid *dispersion* and so that connectors, or other discontinuities in the line, do not cause mode-mixing.

Real transmission lines exhibit resistive and dielectric losses that are usually small enough not to affect most of the previous results, although they can be a subtle cause of mode-mixing. Figure 5 shows the revised 2-port element needed to take account of losses in a fairly realistic way. In this case the same style of derivation that led to equations 8 yields

$$\left(\frac{\partial V}{\partial x}\right) = -IR - L\left(\frac{\partial I}{\partial t}\right) \quad \text{and similarly} \quad \left(\frac{\partial I}{\partial x}\right) = -GV - C\left(\frac{\partial V}{\partial t}\right) \tag{23a,b}$$

which are often referred to as the *Telegraphers' Equations*. The characteristic impedance is now complex

$$\mathbf{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}\,. \tag{24}$$

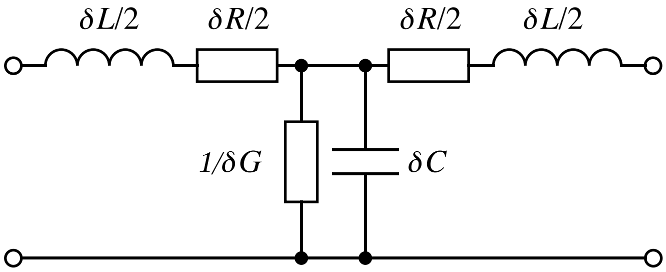

Figure 5. A 2-port network representing a length δ of lossy line.

In real lines the imaginary part of $\mathbf{Z}_0$ is not usually more than about 10% of the real part at 1MHz and, as it decreases at higher frequencies and only affects very long lines at lower frequencies, losses are not often a significant problem. A notable exception is an analogue telephone line which can be kilometres long and operates at kilohertz frequencies. The complex impedance causes dispersion and hence distortion that makes speech unintelligible. If it could be arranged that $RC = LG$ then $\mathbf{Z}_0$ would cease to be frequency dependent and there would be no distortion. Since $RC > LG$ and increasing G would cause unacceptable attenuation the cure is to insert inductors in series with the line at intervals of a fraction of a wavelength. In extreme cases the parameters (*R, L, C and G*) cease to be frequency independent (*e.g.* due to the skin effect) and direct application of Maxwell's equations may be required.

The fact that pulses reflect from sudden changes in the line impedance is the basis for the technique of *time domain reflectometry*, which is used to find faults in long transmission lines. A short pulse is applied to one end of the cable and by timing the arrival of reflections it is possible to learn the position, type (open- or short-circuit) and severity of faults without having to dig up the entire cable first. High-speed computer networks, which rely on transmission lines to carry the data can also be monitored in this way and, with luck and expensive test equipment, faults can often be detected before they start to affect the computers.