The \( \nabla \) Operator

What is a Vector?

If \( S \) and \( S' \) are two sets of Cartesian coordinates sharing a common origin, then vector \( \mathbf{A} \), which has components \( A_x, A_y, A_z \) in the \( S \) frame has components in the \( S' \) frame given by

\[
\begin{bmatrix}
A_{x'} \\
A_{y'} \\
A_{z'}
\end{bmatrix} =
\begin{bmatrix}
a_{xx} & a_{xy} & a_{xz} \\
a_{yx} & a_{yy} & a_{yz} \\
a_{zx} & a_{zy} & a_{zz}
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z
\end{bmatrix},
\]

where the coefficients \( \{a_{ij}\} \) depend only on the orientation of \( S' \) with respect to \( S \). If a quantity transforms in this way then it is a vector, otherwise it is not.

The \( \nabla \) Operator

If we let \( \mathbf{A} = \nabla f \) then provided \( f \) is differentiable

\[
A_x = \frac{\partial f}{\partial x} \quad \text{and} \quad A_{x'} = \frac{\partial f}{\partial x} = a_{xx} \frac{\partial f}{\partial x} + a_{xy} \frac{\partial f}{\partial y} + a_{xz} \frac{\partial f}{\partial z}
\]

with similar result for the other two components so \( \nabla f \) is certainly a vector. We can also see from this that the components of \( \nabla \) which were defined in the \( S \) frame are given in the \( S' \) frame by

\[
\nabla_{x'} = \frac{\partial}{\partial x'} = a_{xx} \frac{\partial}{\partial x} + a_{xy} \frac{\partial}{\partial y} + a_{xz} \frac{\partial}{\partial z} \quad \text{etc.}
\]

so the operator \( \nabla \) transforms in the required way and is, therefore a vector in its own right. However it is also a differential operator and so \( \nabla \) must only be used in ways that satisfy simultaneously the rules for manipulating vectors, and of partial differentiation.