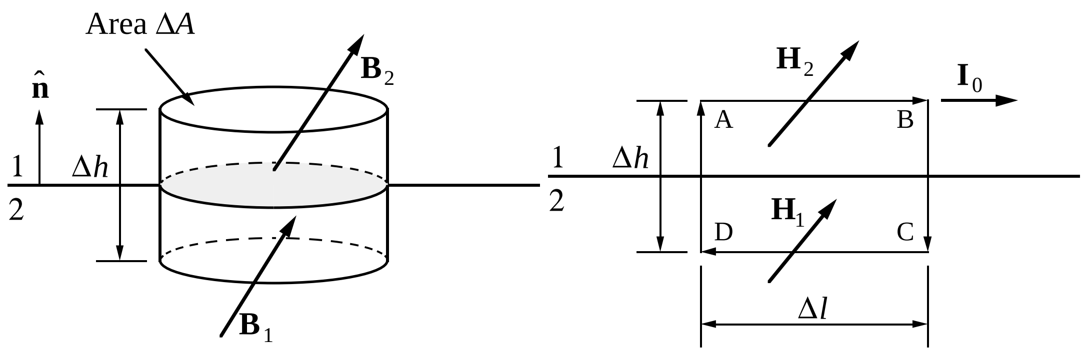
Magnetostatic Boundary Conditions



Consider a Gaussian pill-box at the interface between two different media, arranged as in the figure above. The net enclosed (free) magnetic charge density is zero so as the height of the pill-box Δh tends to zero so the integral form of Gauss's law tells us that

$$\left(\mathbf{B}_2 \cdot \hat{\mathbf{n}}\right)\Delta A - \left(\mathbf{B}_1 \cdot \hat{\mathbf{n}}\right)\Delta A \approx 0$$

which becomes exact in the limit $\Delta A \to 0$ when

$$\left(\mathbf{B}_2 - \mathbf{B}_1\right) \cdot \hat{\mathbf{n}} = 0$$

therefore the component of **B** normal to the interface is continuous.

To find the **H**-field boundary condition we apply Ampère's circuital law to the path ABCD shown in the diagram above. $\mathbf{I}_0$ is the unit vector in the direction AB parallel to the surface so

$$\text{as } \Delta h \to 0 \quad \text{so} \quad \left(\mathbf{H}_2 - \mathbf{H}_1\right) \cdot \mathbf{I}_0 \Delta l = \mathbf{j}_\text{c} \cdot \left(\hat{\mathbf{n}} \times \mathbf{I}_0\right) = \left(\mathbf{j}_\text{c} \times \hat{\mathbf{n}}\right) \cdot \mathbf{I}_0$$

or equivalently

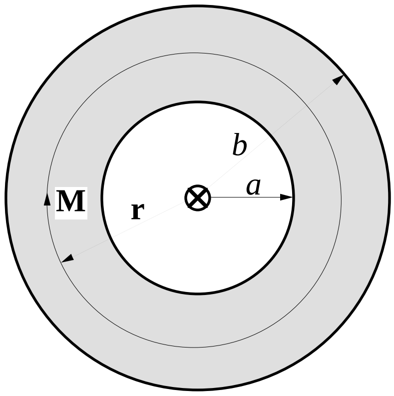
$$\left(\mathbf{H}_2 - \mathbf{H}_1\right)_\parallel = \mathbf{j}_\text{c} \times \hat{\mathbf{n}}.$$

One can take the cross-product of this expression to obtain a form that is useful for deducing $\mathbf{j}_\text{c}$ if **H** is known on each side of the boundary.

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right) = \mathbf{j}_\text{c}.$$

To summarise, the component of **H** tangential to the interface is continuous across the interface unless there is a conduction surface current density $\mathbf{j}_\text{c}$.

Note also that although the flux of **B** is continuous everywhere, the flux of **H** is not.



Example: A long thin straight wire carries a current I and is surrounded by a long coaxial hollow cylinder, inner radius a and outer radius b, of the same paramagnetic sample. Find the magnitude M of the magnetisation inside the sample at distance r from the axis.

Solution: The system is cylindrically symmetric and all the surfaces will be parallel to **H**. The paramagnetic material doesn't have a permanent dipole moment so $\mathbf{M} = M\hat{\boldsymbol{\theta}}$, $\mathbf{H} = M\hat{\boldsymbol{\theta}}$ and

$$\nabla \times \mathbf{H} = \mathbf{j}_{\mathrm{c}} \quad \Rightarrow \quad \oint \mathbf{H} \cdot \mathrm{d}\mathbf{r} = I$$

combined with the symmetry gives

$$\mathbf{H} = \frac{I\hat{\boldsymbol{\theta}}}{2\pi r}.$$

and, defining the susceptibility of the material by $\mathbf{M} = \chi \mathbf{H}$,

$$\mathbf{M} = \frac{\chi I\hat{\boldsymbol{\theta}}}{2\pi r}$$

Note: The cylindrical symmetry simplifies this problem *greatly*. In general finding **B**, **H** and **M** for an arbitrary arrangement of currents and shapes is *very* difficult because the component of **H** normal to the surface will be discontinuous. For example, if the above ring had a thin slot cut radially then to a first approximation **H** would be unchanged from the above results and **B** would be normal to the cut surface and hence equal in the material and the slot. However, the symmetry used to deduce **H** would have been broken and this will affect its values, particularly near the slot, which will change **M** slightly and hence **B** and so on. Numerical methods are the only way to solve most problems accurately.