Curvilinear Coordinate Systems

Many operators have particularly simple forms in Cartesian coordinates and are easy to remember and evaluate. However, when problems deal with highly symmetric systems it is often helpful to use coordinate systems which exploit the symmetry and so it is useful to have a general method for expressing operators in non-Cartesian forms.

Let $f(x,y,z)=u$ specify a surface characterised by the constant parameter u. The family of spherical surfaces of radius u would be $\left(x^2+y^2+z^2\right)^{1/2}=u$ for example. Consider three such surfaces

$$f_1(x,y,z,)=u_1 \qquad f_2(x,y,z,)=u_2 \qquad f_3(x,y,z,)=u_3$$

where the point of intersection is specified by (u_1,u_2,u_3) which are its *orthogonal curvilinear coordinates*. If the surfaces u_1 and $u_1+\mathrm{d}u_1$ are separated by an element of length $\mathrm{d}l_1$ normal to the surface u_1 then $\mathrm{d}l_1=h_1\mathrm{d}u_1$ where $h_1=h_1(u_1,u_2,u_3)$ and $\mathrm{d}l_2$ and $\mathrm{d}l_3$ are similarly defined. For example, the rectangular Cartesian coordinate system is defined by

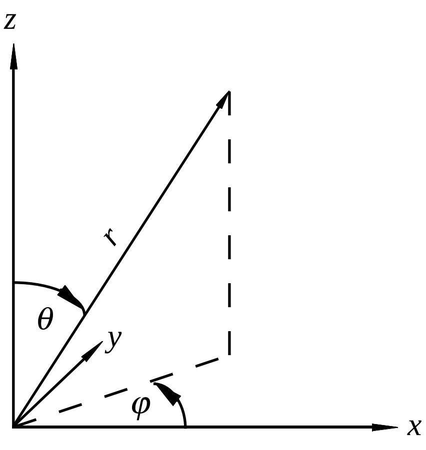
$$f_1(x,y,z,)=x=u_1 \qquad f_2(x,y,z,)=y=u_2 \qquad f_3(x,y,z,)=z=u_3$$

and $h_1=h_2=h_3=1$. The unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ are normal to the u_1, u_2, u_3 surfaces and point towards increasing values of the coordinates with the subscripts assigned an order so that $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2=\hat{\mathbf{e}}_3$. The infinitesimal volume element is therefore

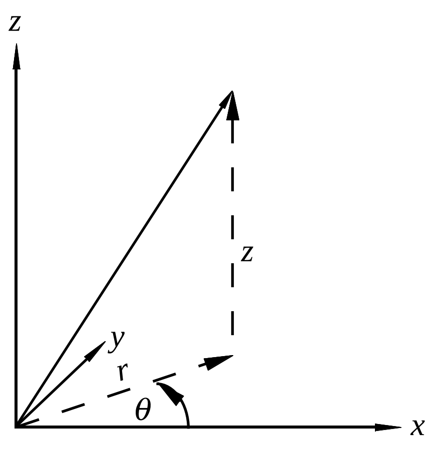
$$dV=\mathrm{d}l_1\mathrm{d}l_2\mathrm{d}l_3=h_1h_2h_3\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3.$$

Note that the Cartesian coordinate system is a special case because, in general, the orientation of the unit vectors vary with the values of u_1, u_2, u_3.

There is more than one conventional way to choose symbols to represent coordinate systems. Unfortuately the choice made in course PHY1016 (*i.e.* (ρ,ϕ,z) for cylindrical and (r,θ,ϕ) for spherical) is not the usual one made in elctromagnetism which uses the symbols ρ and ϕ for charge density and potential respectively. In the following definitions f is a scalar function and $\mathbf{A}=A_1\hat{\mathbf{e}}_1+A_2\hat{\mathbf{e}}_2+A_3\hat{\mathbf{e}}_3$ is a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3. Notice that both r and θ are defined differently in the cylindrical and spherical cases.



Spherical Polar System



Cylindrical Polar System

Coordinate System	(u_1,u_2,u_3)	x	y	z	h_1	h_2	h_3
Cartesian	(x,y,z)	x	y	z	1	1	1
Cylindrical Polar	(r,θ,z)	$r\cos\theta$	$r\sin\theta$	z	1	r	1
Spherical Polar	(r,θ,φ)	$r\sin\theta\cos\varphi$	$r\sin\theta\sin\varphi$	$r\cos\theta$	1	r	$r\sin\theta$

Gradient
$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{\mathbf{e}}_3$$

Divergence
$$\nabla \cdot \mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(h_2h_3A_1\right) + \frac{\partial}{\partial u_2}\left(h_3h_1A_2\right) + \frac{\partial}{\partial u_3}\left(h_1h_2A_3\right)\right]$$

Curl
$$\nabla \times A = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

Laplacian
$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

Laplacian
$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$