Small Current-Loops

Vector potential due to a current loop

The magnetic vector potential at a distant point **r** due to a current I flowing round a small loop C, is found by substituting $\mathbf{J}(\mathbf{r}')\mathrm{d}^3r' \to I\mathrm{d}\mathbf{r}'$ in the definition of the potential so

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_{\mathrm{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

which can be approximated by a series in much the same way as was described on the previous Multipole Expansions handout.

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left(1+\left(\frac{r'}{r}\right)^2 - \frac{2(\mathbf{r}\cdot\mathbf{r}')}{r^2}\right)^{-1/2} \tag{2}$$

and since $r \gg r'$ a simple binomial series will converge rapidly

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[1+\frac{(\mathbf{r}\cdot\mathbf{r}')}{r^2}+\cdots\right] \tag{3}$$

where only terms to first order in r'/r have been retained, an approximation which requires that the origin of the primed coordinate system is close to the centre of the loop. Substituting into equation 1

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint_{\mathrm{C}}\mathrm{d}\mathbf{r}' + \frac{1}{r^3}\oint_{\mathrm{C}}(\mathbf{r}\cdot\mathbf{r}')\mathrm{d}\mathbf{r}' + \cdots\right] \tag{4}$$

and the first term of this obviously vanishes. The second term can be rewritten with the help of a vector identity (VAF-2) which states that

$$(\mathbf{r}'\times\mathrm{d}\mathbf{r}')\times\mathbf{r} = -\mathbf{r}'(\mathbf{r}\cdot\mathrm{d}\mathbf{r}') + \mathrm{d}\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r}) \tag{5}$$

The small change in $\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r})$ due to a small change $\mathrm{d}\mathbf{r}'$ in $\mathbf{r}'$ is

$$\mathrm{d}[\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r})] = \mathbf{r}'(\mathbf{r}\cdot\mathrm{d}\mathbf{r}') + \mathrm{d}\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r}) \tag{6}$$

which is an exact differential. After adding equation 6 to equation 5 we find

$$\mathrm{d}\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r}) = \tfrac{1}{2}(\mathbf{r}'\times\mathrm{d}\mathbf{r}')\times\mathbf{r} + \tfrac{1}{2}\mathrm{d}[\mathbf{r}'(\mathbf{r}'\cdot\mathbf{r})]. \tag{7}$$

The second term vanishes on integration because it is an exact differential and its integral between two points is therefore independent of the path, a closed loop in this case, leaving

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \tfrac{1}{2}(\mathbf{r}' \times \mathrm{d}\mathbf{r}') \times \frac{\mathbf{r}}{r^3} + \cdots \tag{8}$$

so, if the *magnetic dipole moment* **m** of the circuit is defined and the higher order terms are dropped

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \qquad \text{where} \qquad \mathbf{m} = \frac{I}{2} \oint_C \mathbf{r}' \times \mathrm{d}\,\mathbf{r}'. \tag{9}$$

Field due to a current loop

Having found the vector potential its curl can be used to find the field $\mathbf{B}(\mathbf{r})$ due to the current loop described above. We start by using a vector identity (VAF-15)

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\mathbf{m} \times \mathbf{r}}{r^3} \right) = \frac{\mu_0}{4\pi} \left[-(\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3} + \mathbf{m} \left(\nabla \cdot \frac{\mathbf{r}}{r^3} \right) \right] \tag{10}$$

where the two terms in the identity involving $\nabla \cdot \mathbf{m}$ have been dropped because **m** doesn't depend on the coordinates. The first term is best dealt with by writing it out in component form, which in index notation and summing over all indices

$$-(\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3} = -m_j \frac{\partial}{\partial x_j} \frac{x_k \hat{x}_k}{r^3} = -m_j \hat{x}_k \left[x_k \frac{\partial r^{-3}}{\partial x_j} + \frac{1}{r^3} \frac{\partial x_k}{\partial x_j} \right] \tag{11}$$

and therefore

$$-(\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3} = -m_j \hat{x}_k \left[-3 x_k x_j r^{-5} + r^{-3} \delta_{jk} \right] = \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}. \tag{12}$$

The second term of equation 10 is zero, which can be shown by using (VAF-9)

$$\mathbf{m} \left(\nabla \cdot \frac{\mathbf{r}}{r^3} \right) = \mathbf{m} \left(\mathbf{r} \cdot \nabla \left(\frac{1}{r^3} \right) + \frac{1}{r^3} \nabla \cdot \mathbf{r} \right) = \mathbf{m} \left(\mathbf{r} \cdot \frac{-3\mathbf{r}}{r^5} + \frac{3}{r^3} \right) = 0 \tag{13}$$

so, combining all these results

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} \right] \tag{14}$$

which is known as the *magnetic dipole field* because of its similarity to the electric dipole field.

Scalar potential due to a current loop

In regions of space where the current density is zero the curl of the magnetic field must also be zero so it can be described by a magnetic scalar potential ϕ_m

$$\mathbf{B} = -\mu_0 \nabla \phi_\mathrm{m}. \tag{15}$$

Since the divergence of **B** is also zero ϕ_m satisfies Laplace's equation which means that many results derived for electrostatics can be reused for magnetostatics. Things are not entirely straightforward as ϕ_m is often not single valued and getting boundary conditions right can be tricky. A simple example is the scalar potential outside a wire carrying current I, in cylindrical coordinates,

$$\phi_\mathrm{m} = -\frac{I\theta}{2\pi}. \tag{16}$$

By comparing equation 14 with its electrostatic equivalent, the scalar potential of the magnetic dipole moment **m** is

$$\phi_\mathrm{m}(r) = \frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3}. \tag{17}$$

This quantity is useful when calculating the field due to large current-loops, which can be represented as an array of many small loops, and when dealing with problems involving magnetic materials.

Magnetic forces on a small current-loop

To calculate the forces due to an inhomogeneous magnetic field $\mathbf{B}(\mathbf{r})$ on a loop carrying a current I consider a small rectangular loop of side δx and δy lying in the x–y plane. The net force components are

$$\begin{aligned}
F_\mathrm{x} &= I\delta y B_\mathrm{z}(x+\delta x) - I\delta y B_\mathrm{z}(x) = I\delta y \frac{\partial B_\mathrm{z}}{\partial x}\delta x = m\frac{\partial B_\mathrm{z}}{\partial x} \\
F_\mathrm{y} &= I\delta x B_\mathrm{z}(y+\delta y) - I\delta x B_\mathrm{z}(y) = I\delta x \frac{\partial B_\mathrm{z}}{\partial y}\delta x = m\frac{\partial B_\mathrm{z}}{\partial y} \\
F_\mathrm{z} &= -m\left(\frac{\partial B_\mathrm{x}}{\partial x} + \frac{\partial B_\mathrm{y}}{\partial y}\right) = -m(\nabla\cdot\mathbf{B}) + m\frac{\partial B_\mathrm{z}}{\partial z} = m\frac{\partial B_\mathrm{z}}{\partial z}
\end{aligned} \tag{18}$$

where $m\hat{\mathbf{z}}$ is the magnetic dipole moment of the loop. Adding these components together and generalising for a magnetic moment pointing in an arbitrary direction the net force is

$$\mathbf{F} = m\nabla(\mathbf{B}\cdot\hat{\mathbf{m}}) = \nabla(\mathbf{B}\cdot\mathbf{m}) - (\mathbf{B}\cdot\hat{\mathbf{m}})\nabla m. \tag{19}$$

and in the case that m doesn't depend on the position of the loop, this simplifies and becomes

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) = (\mathbf{m} \cdot \nabla)\mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B}) \tag{20}$$

Note: Since an electric field **E** has no curl the equivalent expression for the force on an electrostatic dipole moment **p** is

$$\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E}) = (\mathbf{p} \cdot \nabla)\mathbf{E}. \tag{21}$$

The torque on the loop when it lies in the *x–y* plane has components

$$\Gamma_{\mathrm{x}} = +\delta x I \delta y B_{\mathrm{x}} = +m_{\mathrm{z}} B_{\mathrm{x}} \qquad \Gamma_{\mathrm{y}} = -\delta y I \delta x B_{\mathrm{y}} = -m_{\mathrm{z}} B_{\mathrm{y}} \qquad \Gamma_{\mathrm{z}} = 0 \tag{22}$$

with similar expressions obtained (from symmetric permutations of the coordinates) for the cases when it lies in the *x–z* and *y–z* planes. By considering each of these thre cases as the projection of an arbitrarily orientated loop the results can be summed to obtain the general expression

$$\begin{aligned} \Gamma_{\mathrm{x}} &= m_{\mathrm{y}} B_{\mathrm{z}} - m_{\mathrm{z}} B_{\mathrm{y}} \\ \Gamma_{\mathrm{y}} &= m_{\mathrm{z}} B_{\mathrm{x}} - m_{\mathrm{x}} B_{\mathrm{z}} \qquad \textit{i.e.} \qquad \mathbf{\Gamma} = \mathbf{m} \times \mathbf{B}. \\ \Gamma_{\mathrm{z}} &= m_{\mathrm{x}} B_{\mathrm{y}} - m_{\mathrm{y}} B_{\mathrm{x}} \end{aligned} \tag{23}$$

The results in this section apply to loops of any shape because these can be approximated to arbitrary accuracy by superpositions of smaller square loops.

Potential energy of a current loop

Since the definition of the potential energy V is that it satisfies $\mathbf{F} = -\nabla V$ and equation 20 has exactly this form the potential energy of the current loop is simply

$$V_{\mathrm{m}} = -\mathbf{m} \cdot \mathbf{B} \tag{24}$$

This potential must be used with care as it is *not* the total energy of the current loop because it was derived subject to the assumption that m is constant and this often not be the case, for example if we were to move the loop to a position where **B** was different the Lentz's law current set up would change m, and energy would be needed to counteract this and keep m constant.