Gauss's Law

Coulomb's law states that the electric field $\mathbf{E}(\mathbf{r})$ at a point $\mathbf{r}$ due to a charge Q at another point $\mathbf{r}'$ is

$$\mathbf{E} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

and the field due to a distribution of charge density $\rho(\mathbf{r}')$ is

$$\mathbf{E}(\mathbf{r}) = \int \frac{\rho(\mathbf{r}').(\mathbf{r}-\mathbf{r}')\,\mathrm{d}^3r'}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

because the principle of superposition applies to electrostatic fields. So the flux is

$$\Phi = \oint_{\mathrm{A}} \mathbf{E}\cdot\mathrm{d}\mathbf{A} = \oint_{\mathrm{A}}\left[\int_{\mathrm{V}} \frac{\rho(\mathbf{r}').(\mathbf{r}-\mathbf{r}')\,\mathrm{d}^3r'}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|^3}\right]\cdot\mathrm{d}\mathbf{A}(\mathbf{r})$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\mathrm{V}}\left[\oint_{\mathrm{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{A}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3}\right]\rho(\mathbf{r}')\,\mathrm{d}^3r'.$$

To deal with this, first consider the integral

$$I(\mathbf{r}') = \oint_{\mathrm{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{A}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3}.$$

For all points outside the surface A the integrand is continuous and, as is easily shown by direct differentiation, its divergence vanishes so the divergence theorem can be used to prove that

$$I(\mathbf{r}'\text{ outside A}) = \oint_{\mathrm{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{A}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3} = \int_{\mathrm{V}} \nabla\cdot\left[\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right]\mathrm{d}^3r = 0.$$

However, when $\mathbf{r}'$ lies inside A the singularity at $\mathbf{r}'=\mathbf{r}$ prevents a similiar conclusion. Instead let $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ and

$$\left(\hat{\mathbf{R}}\cdot\mathrm{d}\mathbf{A}\right) = \mathrm{dA_R} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

then

$$\oint_{\mathrm{A}} \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathrm{d}\mathbf{A}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3} = \iint_{\mathrm{A_R}} \frac{\left(\mathbf{R}\cdot\hat{\mathbf{R}}\right)}{R^3}R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi$$

from which we deduce Gauss's law:

$$\Phi = \oint_{\mathrm{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{1}{\varepsilon_0} \int_{\mathrm{V}} \rho(\mathbf{r})\, \mathrm{d}V .$$

Applying the divergence theorem to this result

$$\Phi = \oint_{\mathrm{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \int_{\mathrm{V}} (\nabla \cdot \mathbf{E})\, \mathrm{d}V = \frac{1}{\varepsilon_0} \int_{\mathrm{V}} \rho(\mathbf{r})\, \mathrm{d}V$$

which is true for all volumes so the integrands must be equal. *i.e.*

$$(\nabla \cdot \mathbf{E}) = \frac{\rho}{\varepsilon_0}$$

which is the differential, or 'point', form of Gauss's law. This is a fundamental result and is always true providing *all* the charge is included in the defintion of the charge density.