Ampère's Law

To deduce Ampère's law from the Biot-Savart law we start by defining

$$\mathbf{R} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1}$$

and, since ∇ only operates on *unprimed* coordinates, the curl of the magnetic flux density at point $\mathbf{r}$ is

$$(\nabla\times\mathbf{B}) = \frac{\mu_0}{4\pi}\int_V \nabla\times(\mathbf{j}'\times\mathbf{R})\,\mathrm{d}^3r' \tag{2}$$

where the $\mathbf{j}' = \mathbf{j}(\mathbf{r}')$ is the current density at point $\mathbf{r}'$. The integrand can be expanded with a standard identity (VAF-15)

$$\nabla\times(\mathbf{j}'\times\mathbf{R}) = (\mathbf{R}\cdot\nabla)\mathbf{j}' - (\mathbf{j}'\cdot\nabla)\mathbf{R} + (\nabla\cdot\mathbf{R})\mathbf{j}' - (\nabla\cdot\mathbf{j}')\mathbf{R}. \tag{3}$$

The various terms in this expression are dealt with as follows: $(\mathbf{R}\cdot\nabla)\mathbf{j}' = 0$ and $(\nabla\cdot\mathbf{j}')\mathbf{R} = 0$ because $\mathbf{j}'$ is not a function of unprimed coordinates; since $\nabla\mathbf{F}(\mathbf{r}-\mathbf{r}') = -\nabla'\mathbf{F}(\mathbf{r}-\mathbf{r}')$ the second term can be rewritten $-(\mathbf{j}'\cdot\nabla)\mathbf{R} = (\mathbf{j}'\cdot\nabla')\mathbf{R}$; We have previously showed, by considering Gauss's law applied to a point charge, that

$$\nabla\cdot\mathbf{R} = \nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right) = 4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{4}$$

and therefore

$$(\nabla\times\mathbf{B}) = \mu_0\int_V \mathbf{j}'\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}^3r' + \frac{\mu_0}{4\pi}\int_V (\mathbf{j}'\cdot\nabla')\mathbf{R}\,\mathrm{d}^3r' \tag{5}$$

The first of these integrals is trivial, but the second is dealt with by using another standard identity (VAF-9) to rewrite it as the sum of its components in the *x*-, *y*- and *z*-directions. For example, the *x*-component is

$$\nabla'(\mathbf{j}'R_x) = R_x(\nabla'\cdot\mathbf{j}') + \mathbf{j}'\cdot(\nabla' R_x). \tag{6}$$

The divergence of the current density can be substituted by using the equation of charge continuity expressed in the primed frame

$$(\nabla'\cdot\mathbf{j}') + \frac{\partial\rho'}{\partial t} = 0 \tag{7}$$

where $\rho' = \rho(\mathbf{r}')$ is the charge density, so

$$\mathbf{j}'(\nabla' R_x) = \nabla'\cdot(\mathbf{j}'R_x) + R_x\frac{\partial\rho'}{\partial t} \tag{8}$$

and therefore

$$(\nabla\times\mathbf{B})_{\mathrm{x}} = \mu_0 j_{\mathrm{x}} - \frac{\mu_0}{4\pi}\int_{\mathrm{V}} R_{\mathrm{x}}\frac{\partial\rho'}{\partial t}\mathrm{d}^3r' + \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\nabla'\cdot\left(\mathbf{j}'R_{\mathrm{x}}\right)\mathrm{d}^3r' \tag{9a}$$

$$= \mu_0 j_{\mathrm{x}} - \frac{\mu_0\varepsilon_0}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\int_{\mathrm{V}}\frac{(x-x')\rho'}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r' + \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\nabla'\cdot\left(\mathbf{j}'R_{\mathrm{x}}\right)\mathrm{d}^3r' \tag{9b}$$

$$= \mu_0 j_{\mathrm{x}} + \mu_0\varepsilon_0\frac{\partial E_{\mathrm{x}}}{\partial t} + \frac{\mu_0}{4\pi}\int_{\mathrm{A}}\left(\mathbf{j}'R_{\mathrm{x}}\right)\cdot\mathrm{d}\mathbf{a}' \tag{9c}$$

where the divergence theorem has been used to transform the last term making it clear that it is zero; because V encloses all the currents so $\mathbf{j}'$ must be zero on its boundary A. Summing all three components of curl **B** gives

$$(\nabla\times\mathbf{B}) = \mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{10}$$

which is one of Maxwell's equations and is always true provided, as is the case with all Maxwell's equations, that the implicit space and time variables **r** and *t* are defined with respect to an inertial frame of reference. In the special case when the currents are steady equation 10 simplifies to the *point form of Ampère's law*

$$(\nabla\times\mathbf{B}) = \mu_0\mathbf{j}. \tag{11}$$

Integrating both sides of this expression over a surface A, bounded by a closed path C, and using Stokes's theorem

$$\oint_{\mathrm{C}}\mathbf{B}\cdot\mathrm{d}\mathbf{l} = \oint_{\mathrm{A}}(\nabla\times\mathbf{B})\cdot\mathrm{d}\mathbf{a} = \int_{\mathrm{A}}\mu_0\mathbf{j}\cdot\mathrm{d}\mathbf{a} \tag{12}$$

so when the net current crossing the surface is I

$$\oint_{\mathrm{C}}\mathbf{B}\cdot\mathrm{d}\mathbf{l} = \mu_0 I \tag{13}$$

which is the integral form and is known as *Ampère's circuital law*.