Vector Analysis Formulae

Identities

1. \( (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \)

2. \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \)

3. \( (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \)

4. \( (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{C} \{ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{D}) \} - \mathbf{D} \{ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \} \)

5. \( (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{B} \{ \mathbf{A} \cdot (\mathbf{C} \times \mathbf{D}) \} - \mathbf{A} \{ \mathbf{B} \cdot (\mathbf{C} \times \mathbf{D}) \} \)

6. \( \nabla(fg) = f\nabla g + g\nabla f \)

7. \( \nabla(f/g) = (1/g)\nabla f - (f/g^2)\nabla g \)

8. \( \nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \)

9. \( \nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}) \)

10. \( \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \)

11. \( (\nabla \cdot \nabla)f = \nabla^2 f \)

12. \( \nabla \times (\nabla f) = 0 \)

13. \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \)

14. \( \nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f(\nabla \times \mathbf{A}) \)

15. \( \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} \)

16a. \( \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \)

16b. \( \nabla^2 \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) - \nabla \times (\nabla \cdot \mathbf{A}) \)

17. \( \nabla(1/r) = -\hat{r}/r^2 \)

If \( S \) is the closed surface that encloses the volume \( V \) and \( C \) is the closed curve that bounds an open surface \( A \) then:

18. \( \int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a) \)

19. \( \int_V (\nabla f) \ dV = \int_S f \ dS \)

20. \( \int_V (\nabla \times \mathbf{B}) \ dV = -\int_S \mathbf{B} \times d\mathbf{S} \)

21. \( \int_S \mathbf{B} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{B}) \ dV \) \hspace{1cm} (The Divergence Theorem)

22. \( \int_C \mathbf{B} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{B}) \cdot d\mathbf{A} \) \hspace{1cm} (Stokes’s Theorem)
Special Coordinate Systems

Cartesian Coordinates \((x, y, z)\)

- \(\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}\)
- \(\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\)
- \(\nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}\)
- \(\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\)
- \(\nabla^2 \mathbf{A} = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})\)

Cylindrical Polar Coordinates \((r, \theta, z)\)

- \(\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{\partial f}{\partial z} \hat{z}\)
- \(\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}\)
- \(\nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{r} + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{z}\)
- \(\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\)

Spherical Polar Coordinates \((r, \theta, \phi)\)

- \(\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}\)
- \(\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( A_\theta \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}\)
- \(\nabla \times \mathbf{A} = -\frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( A_\phi \sin \theta \right) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial A_\phi}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial \theta} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}\)
- \(\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}\)