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Abstract. We investigated several proposals utilizing the unique electronic properties of carbon nanotubes (CNTs) for a broad range of applications to THz optoelectronics, including THz generation by Čerenkov-type emitters based on carbon nanotubes and by hot electrons in quasi-metallic nanotubes, frequency multiplication in chiral-nanotube-based superlattices controlled by a transverse electric f eld, and THz radiation detection and emission by armchair nanotubes in a strong magnetic f eld. Dispersion equations of the electron beam instability and the threshold conditions of the stimulated emission have been derived and analyzed, demonstrating realizability of the nanotube-based nanoFEL at realistic parameters of nanotubes and electronic beams.

Keywords: carbon nanotubes, terahertz radiation

1 INTRODUCTION

Creating a compact reliable source of terahertz (THz) radiation is one of the most challenging problems in contemporary applied physics [1]. This great interest in the THz frequency range is fueled by the fact that THz frequencies characterize different important physical processes (rotation of small molecules, collective vibration modes of proteins, absorption in polar liquids, typical superconducting energy gaps, oscillations in gaseous and solid-state plasmas, etc). Reliable THz devices are required for air pollution monitoring, poison gas sensing, DNA manipulation, gene diagnostics, and other applications. Despite the fact that THz technology is at the boundaries of microwave and photonic technology, it is quite underdeveloped compared with the achievements in microwave or photonics. There are very few commercially available instruments for the THz frequency region and very often they lack the precision required for performing accurate measurements. There are also no miniaturized and low-cost THz sources. One of the latest trends is to use single-wall carbon nanotubes (SWNTs) — cylindrical molecules with nanometer diameter and micrometer length [2–4] — as building blocks of novel THz devices [5–7]. In this paper we summarize and discuss several schemes to utilize the physical properties of carbon nanotubes (CNTs) for generation and detection of THz radiation.

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2 CARBON NANOTUBES AS ČERENKOV-TYPE TERAHERTZ EMITTERS

Recently, the idea using the kinetic energy of CNT-guided electron beam for stimulated emission of electromagnetic waves in optical and terahertz ranges has been proposed [8–13]. In this section we present a consistent theory of the effect. There is a wide family of devices utilizing the interaction of electron beams with electromagnetic waves to produce electromagnetic radiation. Started by the invention of klystrons [14], this family embraces such well-known systems as traveling wave tubes (TWT) and backward wave oscillators (BWO) [15], free electron lasers (FEL) [16–19], etc. In such systems, synchronous motion of electrons and electromagnetic wave modulates the electron beam and coherent radiation is produced by electron bunches. The radiation frequency is smoothly tunable due to its dependence on the electron beam energy. Therefore, such types of emitters can operate in a wide spectral range from microwave and infrared frequencies to vacuum ultraviolet (e.g., VUV-FEL at DESY). Several projects aimed at lasing in hard X-ray range have started [20, 21].The synchronization of moving electrons and electromagnetic wave is attained either by slowing down the electromagnetic wave (Čerenkov, Smith-Purcell [22] and quasi-Čerenkov [23] radiation mechanisms) or by applying an external magnetic f eld, which is uniform in gyrotrons [24,25] and spatially periodical in undulators [18]. The oscillator-type mechanism [26] can be also realized for electrons with discrete spectrum of transverse motion (for example, for electron channeling in crystals).

The Čerenkov radiation is governed by the synchronization condition $\omega - \mathbf{ku} = 0$, where $\mathbf{k}$ is the wavevector and $\mathbf{u}$ is the charged particle (electron) velocity. In systems with external f elds the synchronism condition is transformed to $\omega - \mathbf{ku} - \Omega = 0$, with Ω being the electron oscillation frequency. In the oscillator regime, Ω is the transition frequency between electron levels [27]. To achieve coherent generation in the devices described, a high vacuum must be maintained in the region of the electron beam [28]. Otherwise, collisions of electrons with atoms move electrons out of the synchronism and, consequently, lasing is not reached. From this point of view CNTs are unique objects since they exhibit ballistic electrical conduction at room temperature, with mean free paths of the order of microns and even tens of microns [29–31]. Therefore, electrons can emit coherently from the whole CNT length which is typically 1-10 μm. In addition, single- and multi-wall carbon nanotubes can carry a high current density of the order of $10^9 - 10^{10}$ A/cm^2, see e.g. Refs. [32–34]. Lastly, metallic CNTs exhibit a strong slowing down of surface electromagnetic waves (as large as 50-100 times) [35, 36]. Thus, a combination of three CNT key properties:

(i) ballisticity of the electron flow over typical CNT length,
(ii) extremely high current-carrying capacity and
(iii) strong slowing down of surface electromagnetic waves,

makes them suitable candidates for the development of nano-sized Chernekov-type emitters – nano-TWT, nano-BWO and nano-FEL.

2.1 Self–consistent equation of motion for electromagnetic wave and electron beam

Nanotubes – quasi-one-dimensional carbon macromolecules – are obtained by rolling a graphene layer into a cylinder. The transformation can be performed in different manners classif ed by the dual index (n_1, n_2). The two integers n_1 and n_2 represent the vector characterizing the different ways of rolling, with $n_1 = 0$ for zigzag CNTs, $n_1 = n_2$ for armchair CNTs, and $0 < n_1 \neq n_2$ for chiral CNTs. A nanotube can manifest either metallic or semiconductor properties, depending on its radius $R_{\rm cn}$ and how it is rolled. This correlation arises from the transverse quantization of charge carrier motion and is due to the quasi-one-dimensional topology of CNTs [4, 37].

Consider an electron beam moving in an isolated single-wall carbon nanotube oriented along the z-axis. The electron beam can be injected into the nanotube from the outside by an external

source or they can be produced by applying voltage to some section of the nanotube. Accelerated by the voltage, electrons are injected into the working region. Independently on the origin of electrons, their motion in this region is assumed to be ballistic.

As was mentioned above, there is a certain analogy between a CNT guiding electron beam and macroscopic vacuum electron devices. The main (and obvious) distinction is the small cross-sectional radius of CNTs compared to their macroscopic analogs. In CNTs the spatial quantization of the electron motion comes into play and, therefore, classical models for electron beam become inapplicable. The electron motion in CNTs is governed by quantum-mechanical equations. In this paper we shall consider the lasing effect when the generated field is rather large, i.e. the condition

$$E \gg \sqrt{\hbar c}\left(\frac{\omega}{c}\right)^2 \tag{1}$$

is fulfilled [38]. Further, from this place to the section 4, we set $\hbar = 1$. In the case (1) the electromagnetic wave has a classical character and is described by the classical wave equation:

$$\nabla\nabla\cdot\mathbf{E}(\mathbf{r},\omega) - \Delta\mathbf{E}(\mathbf{r},\omega) = \frac{4\pi i\omega}{c^2}\mathbf{j}(\mathbf{r},\omega). \tag{2}$$

If condition (1) does not hold, the number of photons per quantum level becomes too small to apply a classical approach and the electromagnetic field must be considered within the quantum electrodynamics. Quantum-electrodynamical considerations are important in the initial stage of instability development, when a few photons participate in the process. We leave this stage for further analysis focusing on the stage of highly developed instability. Thus, in our model the electron motion is governed by the Schrödinger equation while the electromagnetic field is described by classical Maxwell equations. In the right-hand part of the field equation (2) the quantity $\mathbf{j}(\mathbf{r},\omega)$ is the current density averaged over the quantum states of the electron beam.

The current density in the working region is defined by the well-known equation [39]:

$$\begin{aligned}\mathbf{j}(\mathbf{r},t) = {} & \frac{e}{2m_e}\left\{\psi^*(\mathbf{r},t)\hat{\mathbf{p}}\psi(\mathbf{r},t) - (\hat{\mathbf{p}}\psi^*(\mathbf{r},t))\psi(\mathbf{r},t)\right\} \\ & -\frac{e^2}{m_e c}\left|\psi(\mathbf{r},t)\right|^2\mathbf{A}(\mathbf{r},t).\end{aligned} \tag{3}$$

Here $\hat{\mathbf{p}} = -i\hbar\partial/(\partial\mathbf{r})$ is the momentum operator and $\mathbf{A}(\mathbf{r},t)$ is the vector potential of the electromagnetic field, $\psi(\mathbf{r},t)$ is electron wave function. In what follows we neglect the Fermi law for the electron statistics. This is possible because the number of excited electrons per quantum level is found to be small, even at superior current densities reachable in CNTs [32–34]. Indeed, the number of levels in the interaction volume V is estimated as $\sim Vp^3/(2\pi\hbar)^3$, where p is a typical value of the quasi-momentum of electrons in the beam. The number of electrons in this volume is $\sim n_e V$, where n_e is the electron density. Then, the number of excited electrons per level is given by $\eta_e = (2\pi\hbar)^3 n_e/p^3$. At a current density of 10^8 to 10^{10} A/cm^2 and an excitation energy of the order of several electronvolts, we find $\eta_e \sim 10^{-5} - 10^{-3}$. Therefore, the exchange interaction between electrons in the beam can be neglected.

Let $\psi(\mathbf{r}, t=0) = \psi_n(\mathbf{r})$ be the eigenfunction of an electron not interacting with the electromagnetic wave and moving along the CNT. When the interaction is switched on the wavefunction is represented by the expansion

$$\psi(\mathbf{r},t) = \sum_l a_l(t)\exp(-i\varepsilon_l t)\psi_l(\mathbf{r}) \tag{4}$$

over a complete set of the unperturbed eigenfunctions $\psi_l(\mathbf{r})$ with corresponding energy eigenvalues ε_l. For further convenience, we rewrite the coefficients $a_l(t)$ as $a_l(t) = \delta_{ln} + \delta a_l^{(n)}(t)$,

where δ_{ln} is the Kronecker symbol. The corrections $\delta a_l^{(n)}(t)$ are due to the electron–electromagnetic f eld interaction. Taking into account axial periodicity of the nanotube potential, the wavefunctions $\psi_l(\mathbf{r})$ can be written in accordance with Bloch theorem as

$$\psi_l(\mathbf{r}) = \exp\{ip_l z\} \sum_\tau b_{l\tau} \exp\{i\tau z\}\, u_{l\tau}(\mathbf{r}_\perp). \tag{5}$$

Here p_l is the axial projection of the quasi–momentum of l-th state, $b_{l\tau}$ are constant coeff cients, $\tau = 2\pi q/a$ are the reciprocal lattice constants, a is the CNT spatial period in the axial direction, $u_{l\tau}(\mathbf{r}_\perp)$ are functions dependent only on transverse coordinates and q are integers. The term $\sum\limits_\tau b_{l\tau} \exp\{i\tau z\}\, u_{l\tau}(\mathbf{r}_\perp)$ is periodic in the z direction.

In the linear approximation, the contribution to the electron current originating from the electron–electromagnetic f eld interaction is described by the equation:

$$\begin{aligned}
\delta\mathbf{j}_n(\mathbf{r},t) &= \frac{e}{2m_e}\sum_l \Big\{ \delta a_l^{(n)*}(t) \exp[i(\varepsilon_l-\varepsilon_n)t][\psi_l^*(\mathbf{r})\hat{\mathbf{p}}\psi_n(\mathbf{r}) - (\hat{\mathbf{p}}\psi_l^*(\mathbf{r}))\psi_n(\mathbf{r})] \\
&+ \delta a_l^{(n)}(t) \exp[-i(\varepsilon_l-\varepsilon_n)t][\psi_n^*(\mathbf{r})\hat{\mathbf{p}}\psi_l(\mathbf{r}) - (\hat{\mathbf{p}}\psi_n^*(\mathbf{r}))\psi_l(\mathbf{r})] \Big\} \\
&- \frac{e^2}{m_e c} |\psi_n|^2 \mathbf{A}(\mathbf{r},t).
\end{aligned} \tag{6}$$

Then, applying the standard perturbation–theory technique [39] we obtain the equation describing the dynamics of the coeff cients $\delta a_l(t)$:

$$\begin{aligned}
i\sum \frac{\partial \delta a_l^{(n)}(t)}{\partial t} \psi_l(\mathbf{r}) \exp(-i\varepsilon_l t) = -\frac{e}{2m_e c} \\
\times [\mathbf{A}(\mathbf{r},t)\hat{\mathbf{p}} + \hat{\mathbf{p}}\mathbf{A}(\mathbf{r},t)]\, \psi_n(\mathbf{r}) \exp(-i\varepsilon_n t)\,,
\end{aligned} \tag{7}$$

which is obtained by substituting Eq. (4) into the Schrödinger equation and its subsequent linearization with respect to the electromagnetic f eld strength. The Fourier transform of Eq. (7) gives

$$\begin{aligned}
\delta a_l^{(n)}(\omega) = \frac{e}{2m_e\omega c} \langle l|\, \mathbf{A}\Big(\mathbf{r}, \omega+\varepsilon_l-\varepsilon_n\Big)\hat{\mathbf{p}} \\
+\hat{\mathbf{p}}\mathbf{A}\Big(\mathbf{r}, \omega+\varepsilon_l-\varepsilon_n\Big)\, |n\rangle\ .
\end{aligned} \tag{8}$$

Here we use the standard bra- and ket- notation for the wavefunctions and matrix elements, $|l\rangle = \psi_l(\mathbf{r})$. Only those terms are preserved in Eq. (8) which correspond to the resonant interaction between electrons and electromagnetic f eld. The contribution of the last term in (6) is therefore neglected in Eq. (8). Performing the Fourier transform of Eq. (6) along the axial coordinate and time, we come to the k,ω–space interaction–induced current density correction:

$$\begin{aligned}
\delta\mathbf{j}_n(k,\mathbf{r}_\perp,\omega) = -\frac{e^2}{4m_e^2 c} \sum_{l\tau'\tau} B_{nl}(k,\mathbf{r}_\perp,\omega) \\
\times \Bigg\{ -\frac{b_{l\tau'}^* b_{n\tau}\left[u_{l\tau'}^*\left(\hat{\mathbf{p}}_n+\tau\right)+\left(\hat{\mathbf{p}}_n+\tau\right)u_{l\tau'}^*\right]u_{n\tau}}{\omega+\varepsilon_l\left(p_n-k\right)-\varepsilon_n(p_n)} \\
+\frac{b_{n\tau}^* b_{l\tau'}\left[u_{n\tau}^*\left(\hat{\mathbf{p}}_n+\tau\right)+\left(\hat{\mathbf{p}}_n+\tau\right)u_{n\tau}^*\right]u_{l\tau'}}{\omega+\varepsilon_n(p_n)-\varepsilon_l\left(p_n+k\right)} \Bigg\}\ .
\end{aligned} \tag{9}$$

For convenience, we have introduced the vector form for the lattice constant τ: $\tau = \tau \mathbf{e}_z$, where $\mathbf{e}_z$ is the unit axial vector. The quasi-momentum operator entering the matrix elements is given by $\hat{\mathbf{p}}_n = \{\hat{\mathbf{p}}_\perp, p_n\}$, where the axial components p_n are C-numbers and transverse components $\hat{\mathbf{p}}_\perp$ are operators. These operators act only on the right-adjacent functions. Deriving Eq. (9), we neglected the longitudinal component k of the electromagnetic wave vector in the matrix elements since $\hbar k/p_n \ll 1$. Summation over the lattice constants τ and τ' is not independent: for every τ in sum, the value of τ' must be such that the values $p_n + \tau - \tau'$ are in the f rst Brillouin zone. The coeff cients $B_{nl}(k, \mathbf{r}_\perp, \omega)$ are given by

$$B_{nl}(k, \mathbf{r}_\perp, \omega) = \sum_{\tau'\tau} b_{l\tau'} b^*_{n\tau} \langle u_{n\tau} | (\hat{\mathbf{p}}_n + \tau)\, \mathbf{A}(k, \mathbf{r}_\perp, \omega) + \mathbf{A}(k, \mathbf{r}_\perp, \omega)\, (\hat{\mathbf{p}}_n + \tau) | u_{l\tau'} \rangle \ .$$

Then by substituting Eq. (9) into Eq. (2) we come to a self–consistent f eld equation necessary for the further analysis.

2.2 Dispersion equation for electromagnetic wave coupled with electron beam

The electromagnetic response properties of an isolated single-wall CNT were studied in Ref. 35 on the base of a tight-binding microscopic model of the CNT conductivity and the effective boundary conditions for an electromagnetic f eld imposed on the CNT surface. A detailed analysis of the eigenwave problem has revealed that there are strongly slowed down surface waves propagating in CNTS leading to the concept of nanotubes as surface-wave nanowaveguides. Considering the electron beam as a perturbation, we can use the dispersion equation for the surface waves and the propagation constants obtained in Ref. 35 as a zero–order approximation. Then, the self-consistent f eld of the electromagnetic wave coupled with an electron beam can be presented by the expansion

$$\mathbf{A}(k, \mathbf{r}_\perp, \omega) = \sum_m \alpha_m(k, \omega) \mathbf{A}_m(\mathbf{r}_\perp)\,, \tag{10}$$

where the vector potentials $\mathbf{A}_m(\mathbf{r}_\perp)$ correspond to the electromagnetic f eld eigenfunctions evaluated in Ref. 35 and $\alpha_m(k, \omega)$ are the coeff cients to be found. Substitution of Eqs. (10), (9) and (6) into (2) gives a system of equations for the electromagnetic f eld interacting with the electrons occupying the n-th state:

$$\sum_m \left(k^2 - k_m^2\right) \alpha_m(k, \omega) \mathbf{A}_m(\mathbf{r}_\perp) = -\frac{4\pi}{c} \frac{e^2 n_e}{4 m_e^2 c} \sum_{l\tau'\tau} B_{nl}(k, \mathbf{r}_\perp, \omega)$$

$$\times \left\{ -\frac{b^*_{l\tau'} b_{n\tau} \left[u^*_{l\tau'} \left(\hat{\mathbf{p}}_n + \tau\right) + \left(\hat{\mathbf{p}}_n + \tau\right) u^*_{l\tau'}\right] u_{n\tau}}{\omega + \varepsilon_l \left(p_n - k\right) - \varepsilon_n(p_n)} \right.$$

$$\left. + \frac{b^*_{n\tau} b_{l\tau'} \left[u^*_{n\tau} \left(\hat{\mathbf{p}}_n + \tau\right) + \left(\hat{\mathbf{p}}_n + \tau\right) u^*_{n\tau}\right] u_{l\tau'}}{\omega + \varepsilon_n(p_n) - \varepsilon_l \left(p_n + k\right)} \right\}. \tag{11}$$

Here k_m are the wavenumbers corresponding to the physical system devoid of the electron beam. As one can see, in deriving Eq. (11) we have proceeded from the single–electron dynamics to the dynamics of the electron beam: n_e is the electron density. Multiplying left- and right–hand parts of Eq. (11) by $\mathbf{A}^*_m(\mathbf{r}_\perp)$ and utilizing the wavefunctions' orthogonality, we come to the dispersion equation as follows:

$$k - k_m = -\frac{\omega_L^2}{8k_m m_e c^2} \sum_l |B_{nl}^{(m)}|^2 \times \left[\frac{1}{-\omega + \varepsilon_n(p_n) - \varepsilon_l(p_n - k)} + \frac{1}{\omega + \varepsilon_n(p_n) - \varepsilon_l(p_n + k)} \right]. \quad (12)$$

The upper index in $B_{nl}^{(m)}$ relates the matrix element with the corresponding mode of the electromagnetic f eld $\mathbf{A}_m(\mathbf{r}_\perp)$; $\omega_L = 2\sqrt{\pi e^2 n_e / m_e}$ is the Langmuir frequency of the electron beam. The transcendent dispersion equation (12) predicts the existence of a variety of branches of wavenumber k. Among them, the number of branches to be accounted for is def ned by specif c physical parameters of the analyzed system. In the vicinity of the resonance, only terms corresponding to the resonant interaction, one or several (in the case of degeneracy), can be kept in the dispersion equation. If the difference between levels exceeds the linewidth, only the resonant term is of importance.

2.3 Classical and quantum limits in synchronism conditions

The two terms in the right-hand side of Eq. (12) dictate two synchronism conditions corresponding to the resonant interaction between the electron beam and electromagnetic wave:

$$\pm\omega + \varepsilon_n(p_n) - \varepsilon_l(p_n \pm k) = 0. \quad (13)$$

The plus and minus signs in Eq. (13) correspond to the absorption and the emission of a photon by an electron, respectively. Depending on the relation between electron and photon energies, different interaction regimes are realized. As we restrict ourselves to the case when the photon momentum is much less than the electron one, the electron energy $\varepsilon_l(p_n \pm k)$ can be presented by the truncated Taylor series as

$$\varepsilon_l(p_n \pm k) = \varepsilon_l(p_n) \pm k \frac{\partial \varepsilon_l(p_n)}{\partial p_n} \equiv \varepsilon_l(p_n) \pm k\, v_l\,, \quad (14)$$

where v_l is the electron group velocity. Then, the denominators in (12) can be represented by

$$\pm\omega + \varepsilon_n(p_n) - \varepsilon_l(p_n \pm k) \approx \pm(\omega - k v_l \pm \Omega_{nl}) + \frac{1}{2}\frac{\partial^2 \varepsilon_l}{\partial p_n^2} k^2\,. \quad (15)$$

The f rst term in the right-hand side of this equation is analogous to the standard term $\omega - ku \pm \Omega$ in the synchronism condition [26]. The only difference is that the velocity of the free electrons is replaced by the group velocity of quasi-electrons, v_l, and the undulation frequency is replaced by the transition frequency, $\Omega_{nl} = \varepsilon_n(p_n) - \varepsilon_l(p_n)$, between the CNT energy bands. The last term in Eq. (15) originates from the quantum recoil of an electron during the emission (absorption) of a photon and induces a red (blue) shift in the transition frequency. This term is inversely proportional to the electron effective mass (second derivative of the energy). Let $l = s$ be an electron level corresponding to the resonant interaction. Then, within the approximation stated, the dispersion equation takes the following form:

$$k - k_m = \frac{2 b_{ns}^{(m)} \left(\frac{k^2}{2} \frac{\partial^2 \varepsilon_s}{\partial p_n^2} - \Omega_{ns} \right)}{(\omega - k v_s)^2 - \left(\frac{k^2}{2} \frac{\partial^2 \varepsilon_s}{\partial p_n^2} - \Omega_{ns} \right)^2}\,, \quad (16)$$

where

$$b_{ns}^{(m)} = -\frac{\omega_L^2}{8 m_e k'_m c^2} |B_{ns}^{(m)}|^2, \;\; k'_m = \mathrm{Re}(k_m).$$

In the case of intraband transitions $\Omega_{ns} = 0$ and Eq. (16) takes the form of the dispersion equation for the instability which takes into account recoil [16].

Depending on the ratio between the radiation linewidth and the recoil-induced detuning, two different generation regimes are realized. In the low–gain limit [18] the spontaneous emission linewidth can be estimated as $\Delta\omega/\omega \sim c/(\omega L)$, where L is the interaction length. If the linewidth exceeds the recoil energy, the recoil term in the denominator of Eq. (16) can be neglected and the classical interaction regime is realized. The dispersion equation in that case takes the traditional form of the second–order Čerenkov resonance:

$$k - k_m = k^2 \frac{\partial^2 \varepsilon_s}{\partial p_n^2} \frac{b_{ns}^{(m)}}{(\omega - k v_s)^2} \,. \tag{17}$$

The *spatial increment* of the instability $k'' = \mathrm{Im}(k)$ can be estimated using the method of weakly coupled modes [40]. According to this method, the interaction between the electromagnetic wave and the electron beam is essential only in the vicinity of the point $(\omega_0, k_0 = \omega_0/v_s)$ where the dispersion curves of the noninteracting modes, $\omega - k v_s = 0$ and $k(\omega) = k_m(\omega)$, are crossed. Then k_m is represented by the expansion

$$k_m(\omega) = k_0 + \left.\frac{\partial k_m(\omega)}{\partial \omega}\right|_{\omega=\omega_0} (\omega - \omega_0) \,. \tag{18}$$

Substituting this expansion and $k = k_0 + \Delta k$ into Eq. (17) results in a third–order algebraic equation with respect to Δk. From this equation, the instability spatial increment is estimated at the frequency $\omega = \omega_0$ as

$$|\Delta k''| = \frac{\sqrt{3}}{2} \left| b_{nn}^{(m)} \frac{\partial^2 \varepsilon_n}{\partial p_n^2} \frac{k^2}{v_n^2} \right|^{1/3} , \tag{19}$$

where $\Delta k'' = \mathrm{Im}(\Delta k)$. Since $b_{nn} \sim n_e$, the increment is found to be the 3-rd root of the electron density. Such a dependence is typical for the Compton-type radiative instability [18].

In the opposite case, when the linewidth is less then the difference between the emission and the absorption frequencies, we fall into the regime of *strong quantum recoil impact*. In this case, only the term corresponding to the emission survives in the dispersion equation (12), which therefore is reduced to

$$k - k_m = b_{nn}^{(m)} \frac{1}{\omega - v_s k - \frac{1}{2} \frac{\partial^2 \varepsilon_n}{\partial p_n^2} k^2} . \tag{20}$$

As a result, the instability increment is given by

$$|\Delta k''| = \left| \frac{b_{nn}^{(m)}}{v_n} \right|^{1/2} , \tag{21}$$

i.e., turns out to be proportional to the square root of the electron density.

Next we present a detail discussion of the different generation regimes and give some numerical estimates of physical parameters corresponding to these regimes.

2.4 Boundary conditions for a finite–length nanotube

In Secs. 2.2 and 2.3, dispersion equations have been derived providing us with wavenumber eigenvalues in an inf nite-length CNT guiding an electron beam. As a next step, edge conditions must be imposed upon the system to account for the f nite length of the interaction zone.

These conditions are stated as the requirement for the perturbations of the electron and current densities, generated by the electron beam – electromagnetic wave interaction, to be zero at the input of the working zone, i.e.

$$\delta n_e(z=0) = \delta j_n(z=0) = 0\,. \tag{22}$$

The condition that the tangential electric f eld component and the axial component of the magnetic f eld be continuous on the CNT surface yields an additional boundary condition. We write it in the simplif ed form [41] as

$$E(z=0) = \alpha E(z=L)\,, \tag{23}$$

where α is the ref ection coeff cient of electromagnetic f eld from the working zone boundaries.

The f eld distribution in a f nite-length system consisting of several parts can be found by solving electrodynamical problem in each region separately and then joining the solutions by means of the boundary conditions. In the interaction region, the electromagnetic f eld is given by

$$E(z) \sim \sum_{i=1}^{N} c_i \exp\bigl(ik^{(i)}z\bigr)\,, \tag{24}$$

where the summation is performed over all electromagnetic modes in the CNT; the wavenumbers $k^{(i)}$ are determined by the corresponding dispersion equations. Note that the ref ection of the electromagnetic waves from the boundaries back into the working zone creates positive feedback in the system and thus allows the accumulation of electromagnetic energy and provides an oscillator regime.

2.5 Starting current at a large quantum recoil

In the quantum interaction regime, when the quantum recoil exceeds the linewidth, the instability is described by the quadratic dispersion equation (20) with solutions $k^{(1)}$ and $k^{(2)}$. Consequently, the electric f eld and the perturbation of the current density in the working zone are given by

$$E \sim c_1 \exp\bigl(ik^{(1)}z\bigr) + c_2 \exp\bigl(ik^{(2)}z\bigr)\,, \tag{25}$$

$$\delta j_n \sim \frac{c_1}{\delta_1} \exp\bigl(ik^{(1)}z\bigr) + \frac{c_2}{\delta_2} \exp\bigl(ik^{(2)}z\bigr)\,. \tag{26}$$

The coeff cients

$$\delta_{1,2} = 1 - \frac{v_n}{\omega} k^{(1,2)} + \frac{1}{2\omega}\,\frac{\partial^2 \varepsilon_n}{\partial p_n^2}\, k^{(1,2)2} \tag{27}$$

introduce deviations of the wavenumbers $k^{(1)}$ and $k^{(2)}$ from the synchronism, and the coeff cients c_i are determined from the boundary conditions as was discussed in Sect. 2.4. Using the boundary conditions (22) and (23), we arrive at the linear system for c_i as follows:

$$\begin{aligned} &c_1 + c_2 = \alpha\bigl[c_1 \exp\bigl(ik^{(1)}L\bigr) + c_2 \exp\bigl(ik^{(2)}L\bigr)\bigr]\,, \\ &\frac{c_1}{\delta_1} + \frac{c_2}{\delta_2} = 0\,. \end{aligned} \tag{28}$$

The nontrivial solution of this system is determined by the equation

$$\delta_1\bigl[1 - \alpha \exp\bigl(ik^{(1)}L\bigr)\bigr] - \delta_2\bigl[1 - \alpha \exp\bigl(ik^{(2)}L\bigr)\bigr] = 0\,. \tag{29}$$

The current density satisfying Eq. (29) is the *threshold current density* of the generation. To evaluate this quantity, the characteristic equation (29) must be solved together with Eq. (20). Substituting the roots

$$k^{(1,2)} = k_{m,\mathrm{ch}} + \frac{b_{nn}^{(m)}}{v_n(k_{\mathrm{ch}} - k'_m)} \tag{30}$$

of the dispersion equation (20), with k_{ch} extracted from the synchronism condition $\omega - k_{\mathrm{ch}}v_n + (k_{\mathrm{ch}}^2/2)\partial^2\varepsilon_n/\partial p_n^2 = 0$, into Eq. (29) and solving the resulting equation with respect to the current density, we obtain

$$\frac{b_{nn}^{(m)}}{v_n} L^2 \frac{\sin^2 x}{x^2} = 1 - |\alpha| + Lk''_m\,, \tag{31}$$

where

$$x = \left(\omega - k'_m v_n + \frac{k_m'^2}{2}\frac{\partial^2\varepsilon_n}{\partial p_n^2}\right)\frac{L}{2c} \tag{32}$$

is the dimensionless off–synchronism parameter.

Physically, Eq. (31) establishes the energy balance in the working zone. Its left-hand side determines the radiation production, which is proportional to the electron density n_e and to the squared interaction length. The factor $\sin^2 x/x^2$ determines the so called *gain curve* — the gain dependence on the off-synchronism parameter x. In the case considered the gain curve is symmetrical with respect to $x = 0$ and is maximal at zero deviation x. Further we compare this result with the classical case of small recoil and demonstrate a signif cant difference in the behavior of gain curves. The term $1 - |\alpha|$ in the right-hand side of Eq. (31) corresponds to the radiation leakage through the boundaries of the interaction zone while the last term specif es the radiation absorption by the nanotube.

The energy balance equation (31) allows the evaluation of the threshold current density. If the current density in CNT exceeds the threshold value, the generation process occurs. The characteristic time of the instability development is inversely proportional to the *absolute instability increment* $\omega'' = \mathrm{Im}(\omega)$, which is derived by solving the generation equation (29) with respect to $\omega(k)$. In the low-gain regime [18], which implies the conditions $|\Delta k''|L \ll 1$ and $1 - \alpha \ll 1$, the increment is given by:

$$\omega''_m = \left[\frac{\partial k_m}{\partial\omega}\right]^{-1}\left(\frac{b_{nn}^{(m)}}{v_n} L \frac{\sin^2 x}{x^2} - \frac{1 - |\alpha|}{L} - k''_m\right). \tag{33}$$

In the linear stage of the radiative instability development, the electromagnetic f eld grows with time as $\exp(\omega''_m t)$.

2.6 Starting current in the classical regime of interaction

In the case when quantum recoil can be neglected, the dispersion equation (17) has three roots

$$k^{(1)} = k_m - b_{nn}^{(m)}\frac{\partial^2\varepsilon_n}{\partial p_n^2}\frac{k_m'^2}{(\omega - v_n k'_m)^2}$$

$$k^{(2,3)} = k_{\mathrm{ch}} \pm \frac{i}{v_n}\sqrt{b_{nn}^{(m)}\frac{\partial^2\varepsilon_n}{\partial p_n^2}\frac{k_m'^2}{k_{\mathrm{ch}} - k'_m}}. \tag{34}$$

and, consequently, the electromagnetic f eld in the interaction region is given by Eq. (24) with $N = 3$. Correspondingly, perturbations of the electron and the current densities in the beam are